A tale of two hotels

Based on a true story

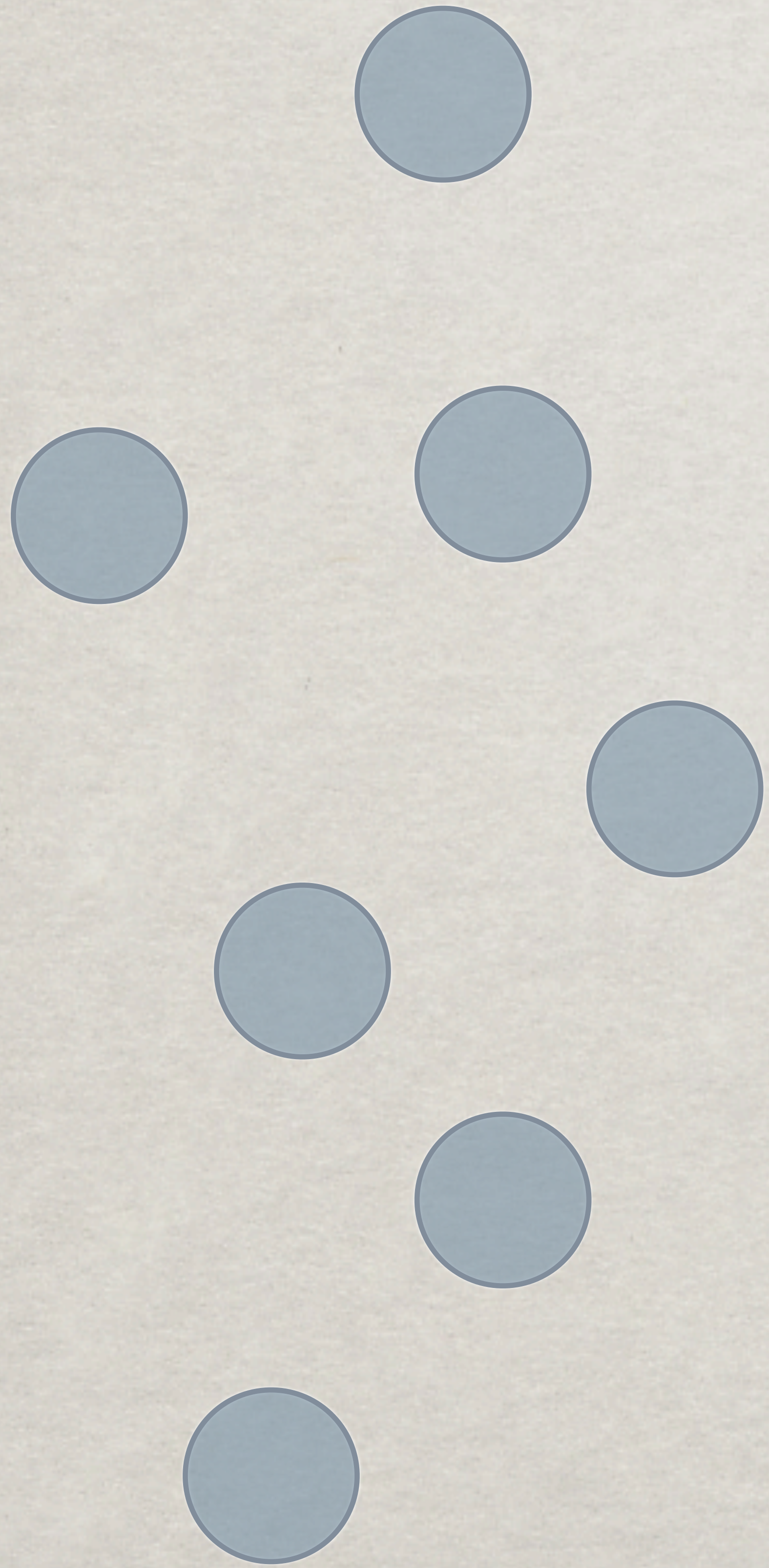
by Prof. B V Rao

GADADHAR MISRA

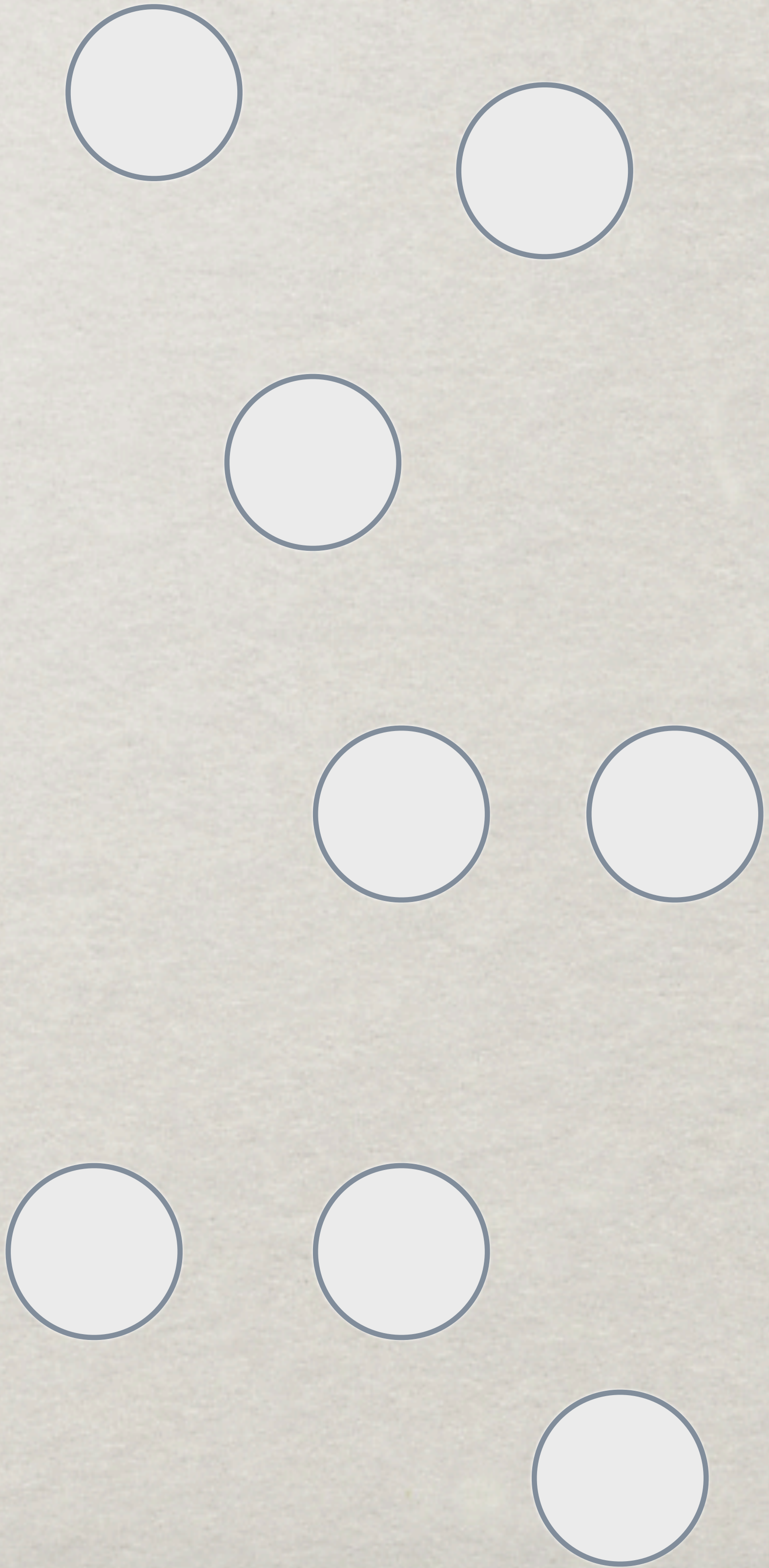
For some time,
let us forget that we know how to count.

But now you want to know if
you got more pens than your friend.

How do you compare,
without counting?



Rahul
v/s
Sachin



Rahul
v/s
Sachin

Rahul
v/s
Sachin

Rahul
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Rahul
v/s
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Rahul

v/s

Sachin

Rahul
v/s
Sachin

Rahul
v/s
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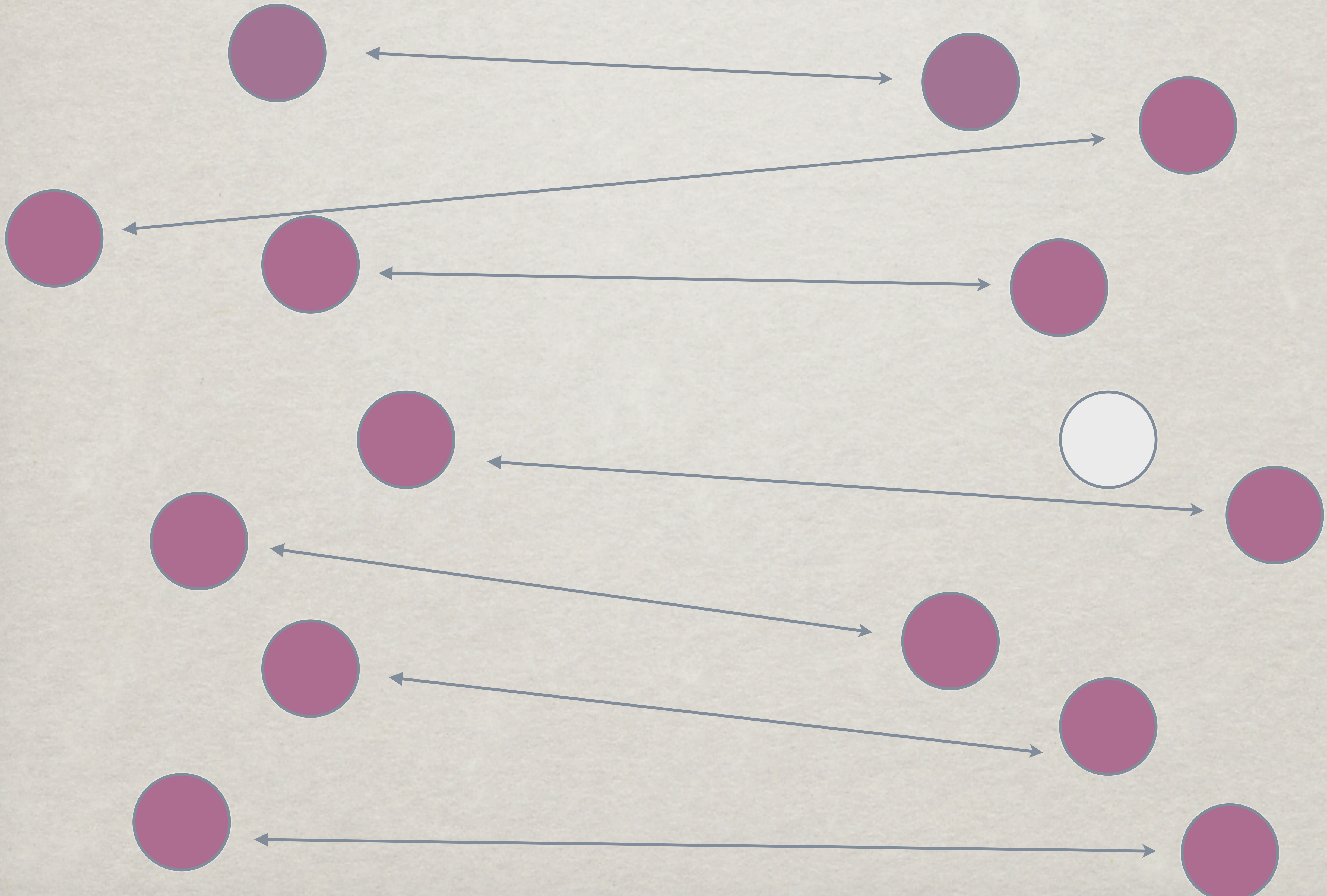
Rahul
v/s
Sachin

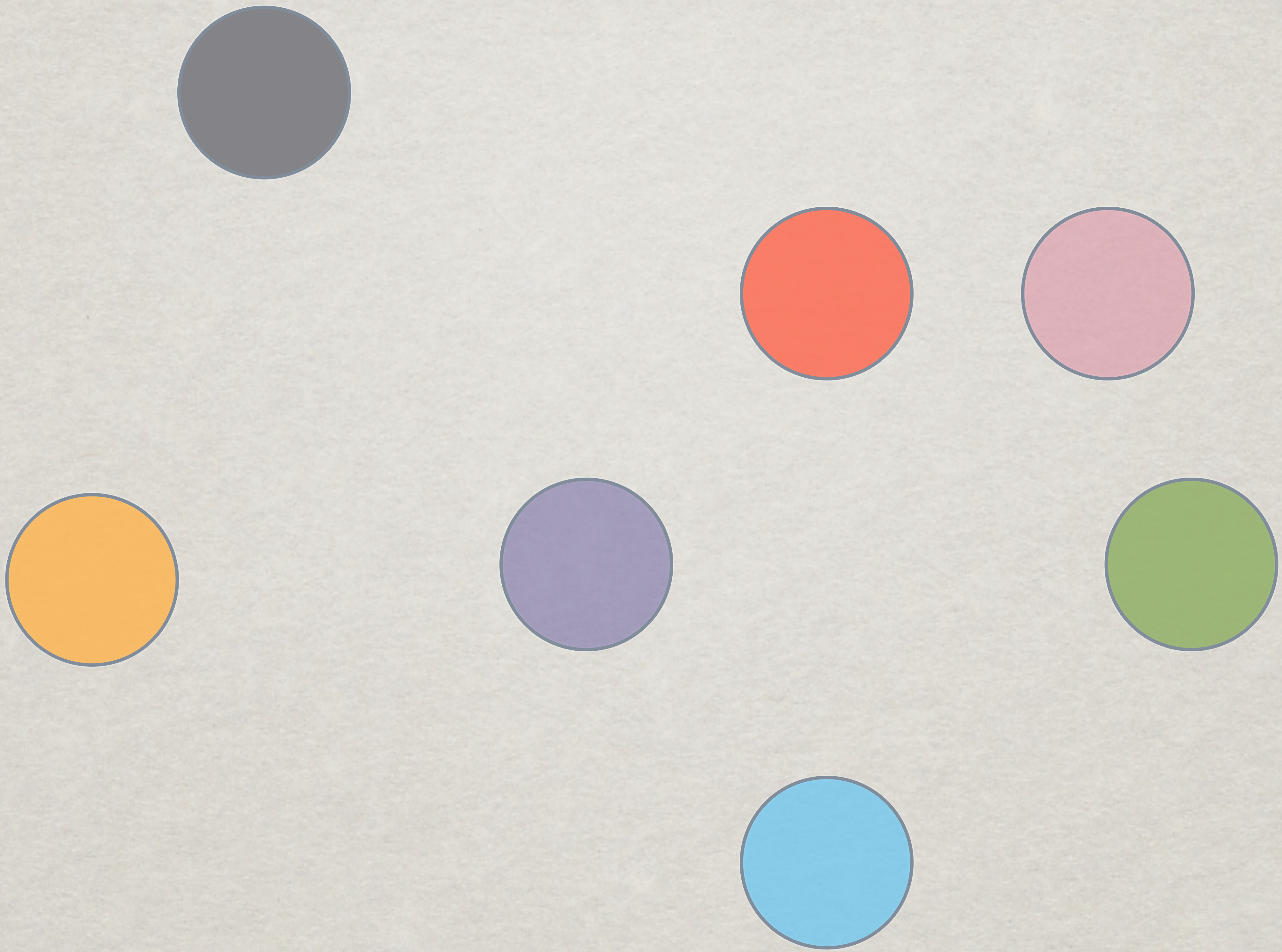
Rahul
v/s
Sachin

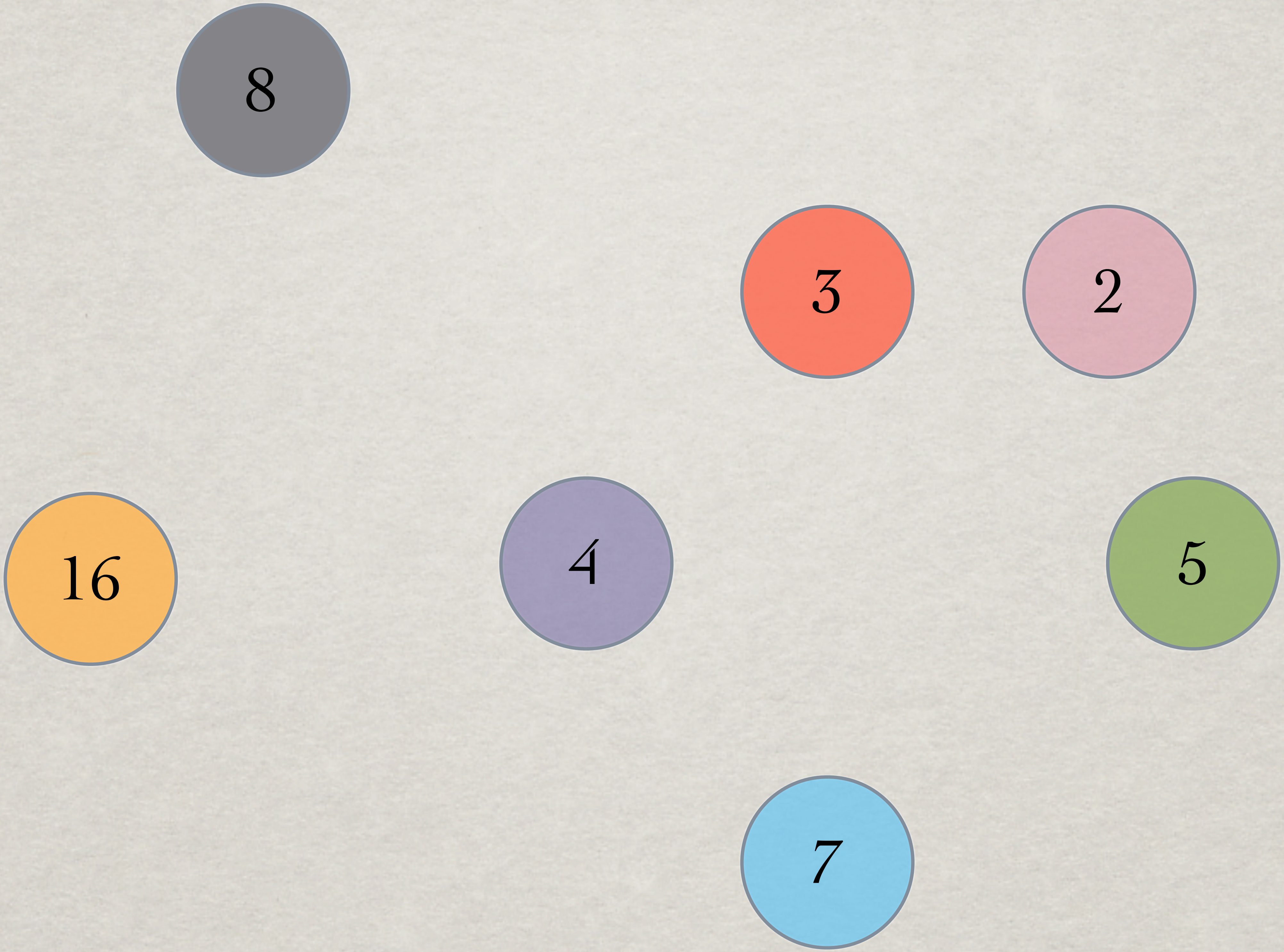
Note!

We managed to compare the sizes of two sets even without knowing anything about numbers!



One-to-one correspondence

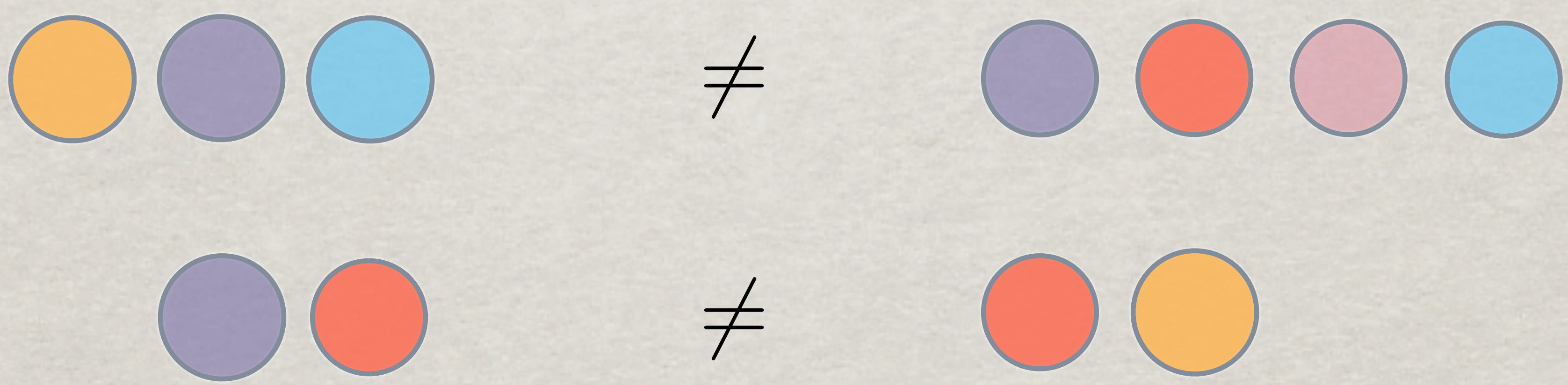


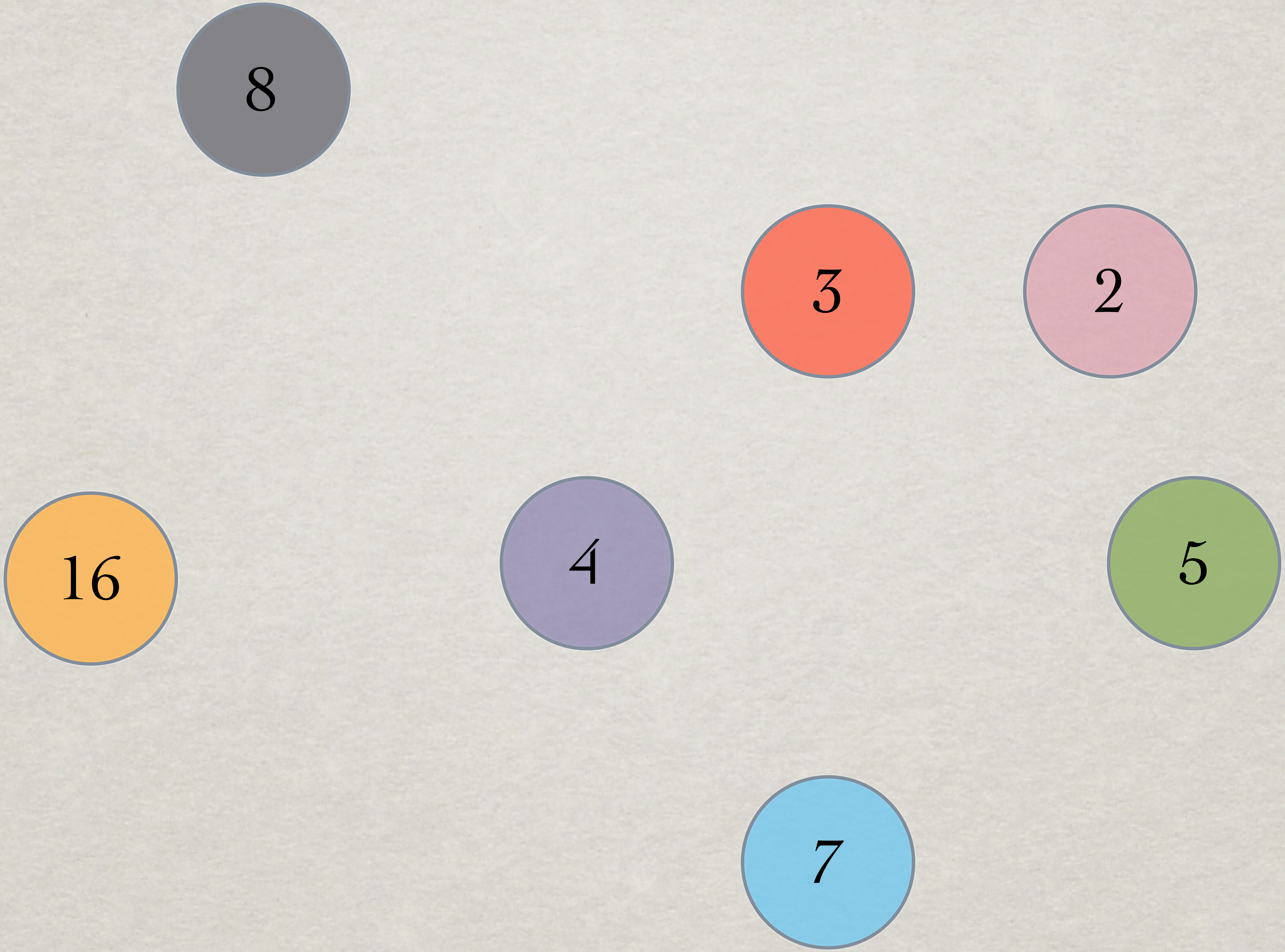

8
3
2
16
4
5
7

I will add these numbers
to find the “value” of your set.

But I also want to be sure that…

if two of you have picked different
sets of colors,
then the values of your sets are different too.

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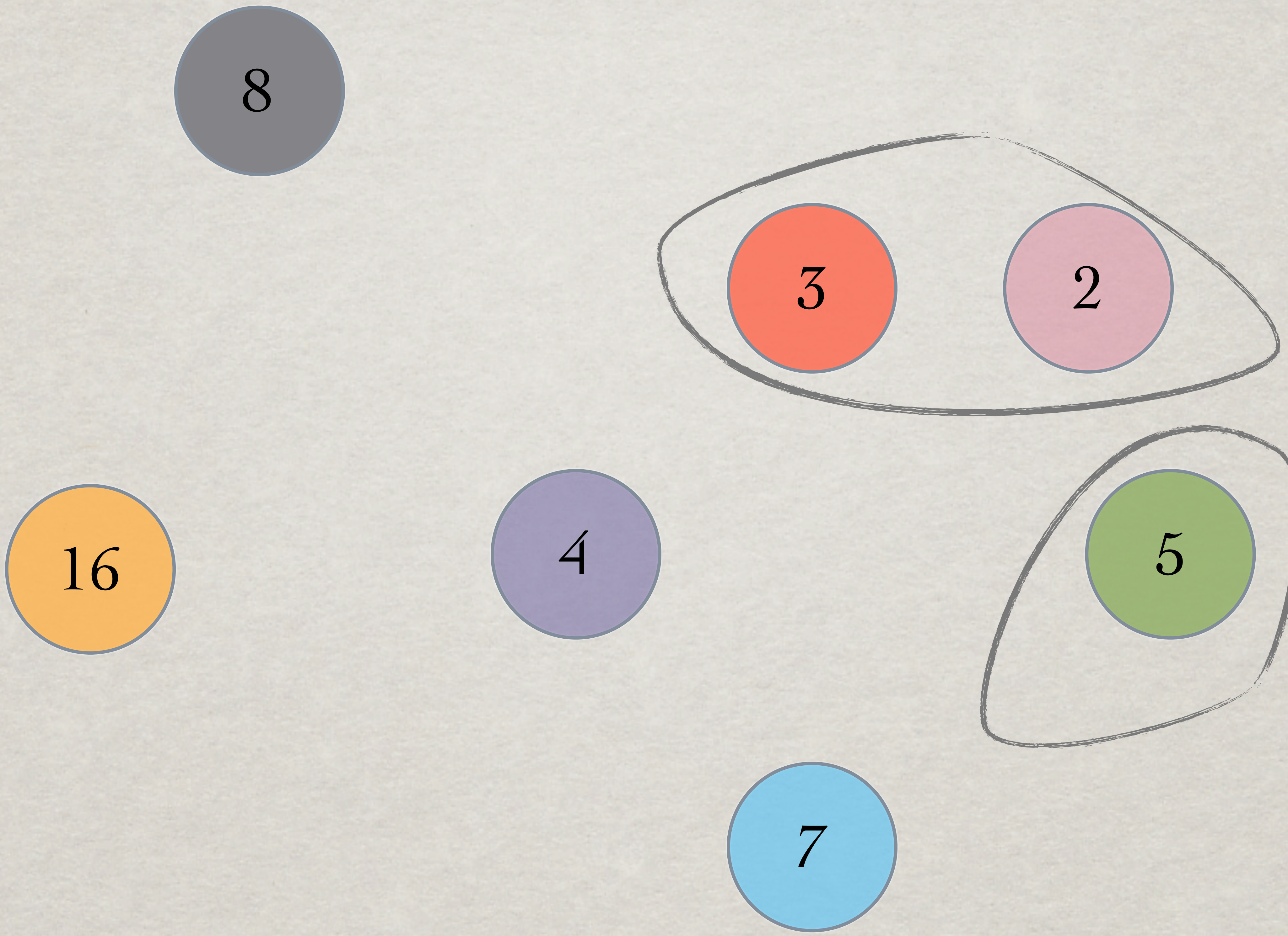
3
2
16
4
5
7


8
3
2
16
4
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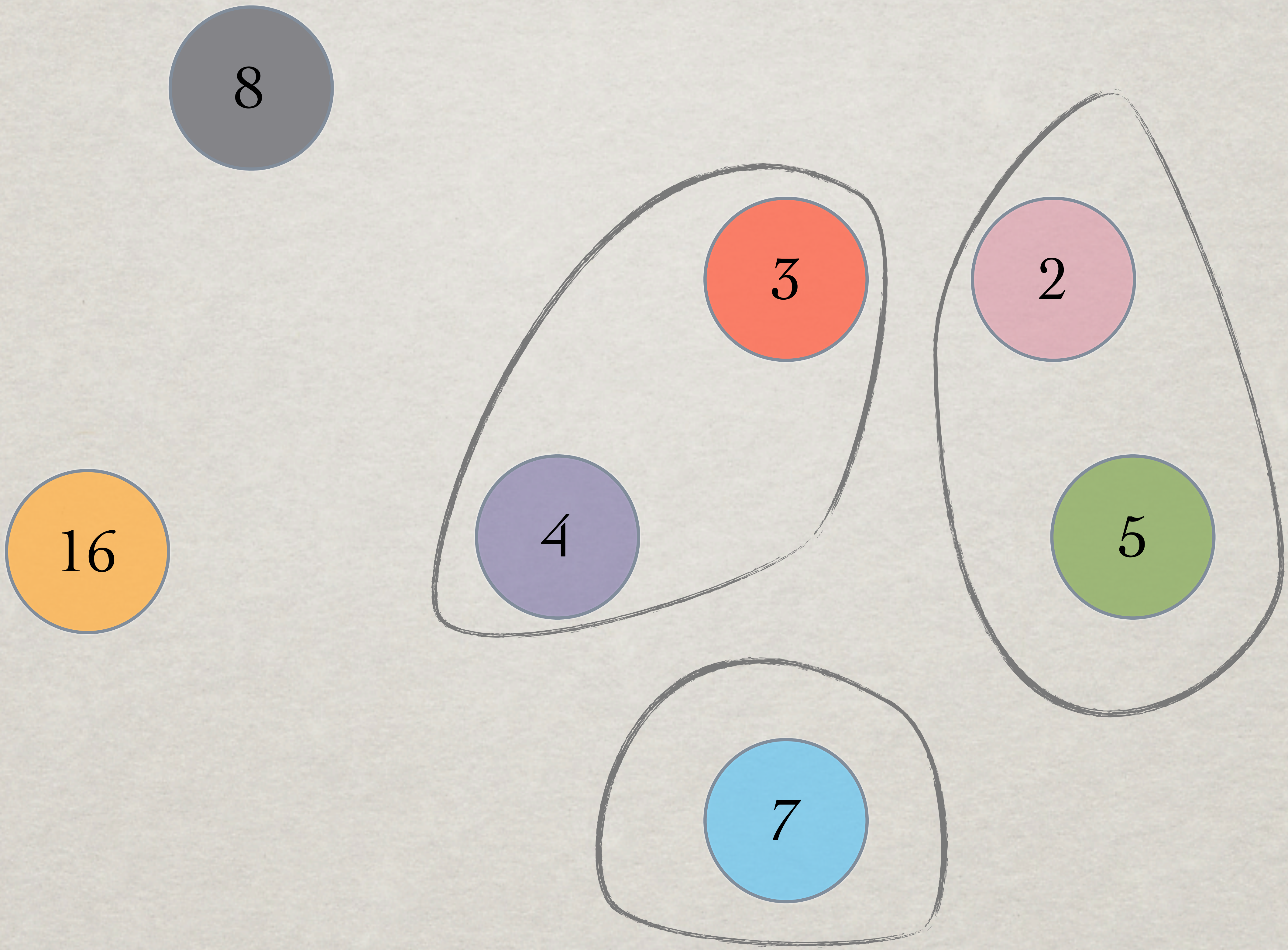
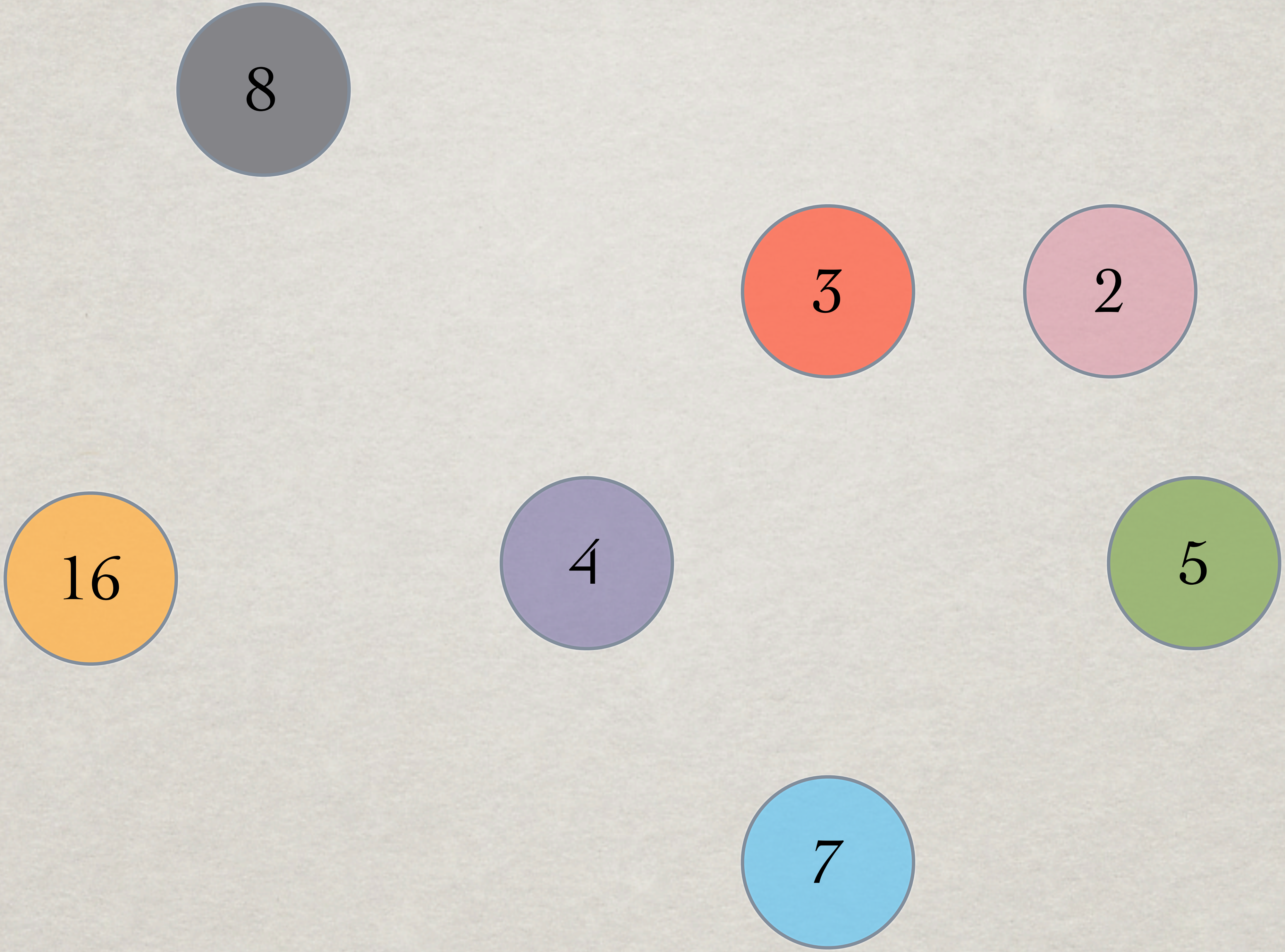
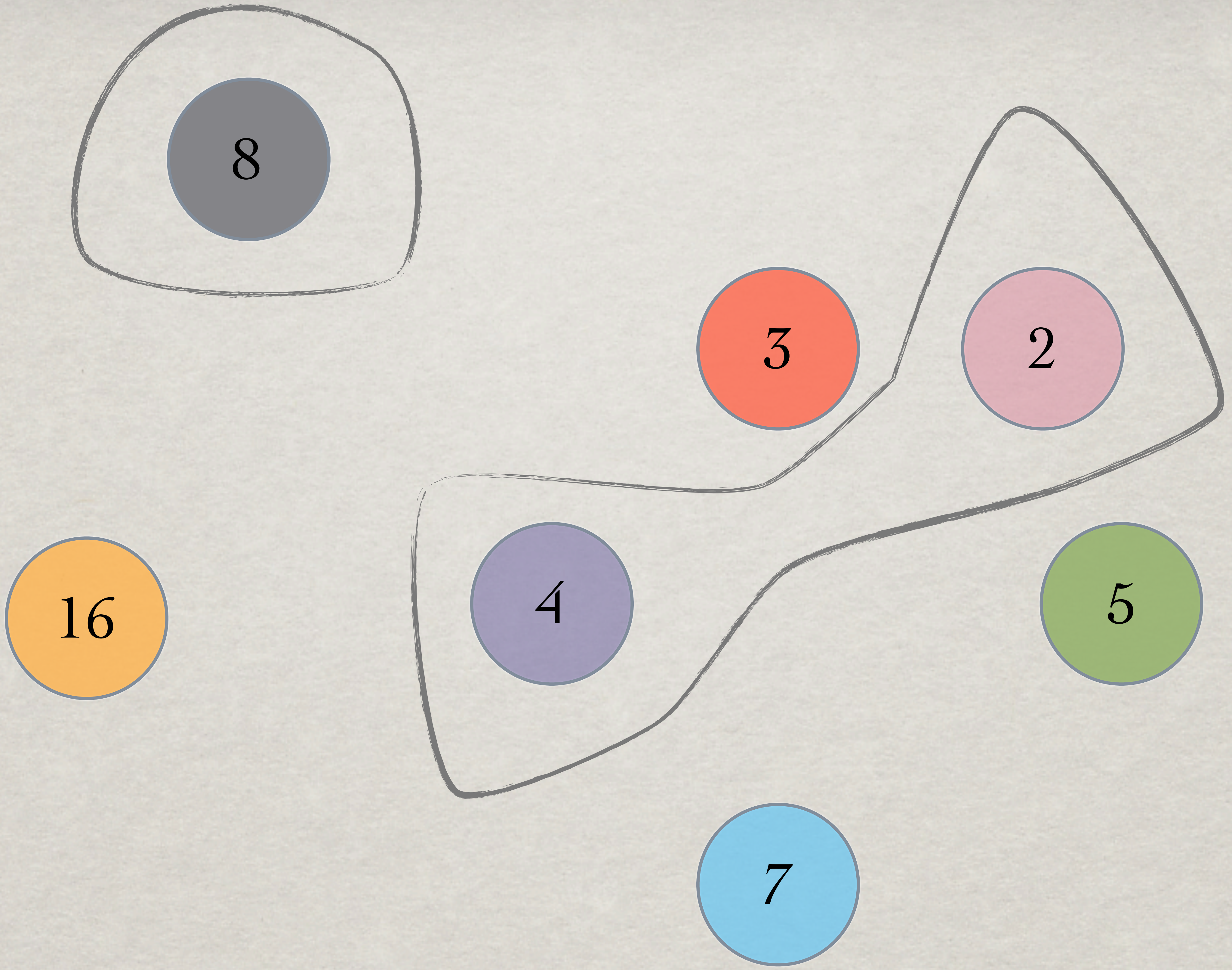

8
3
2
16
4
5
7

Okay, let's try
multiplying these numbers instead.


8
3
2
16
4
5
7


8
3
2
16
4
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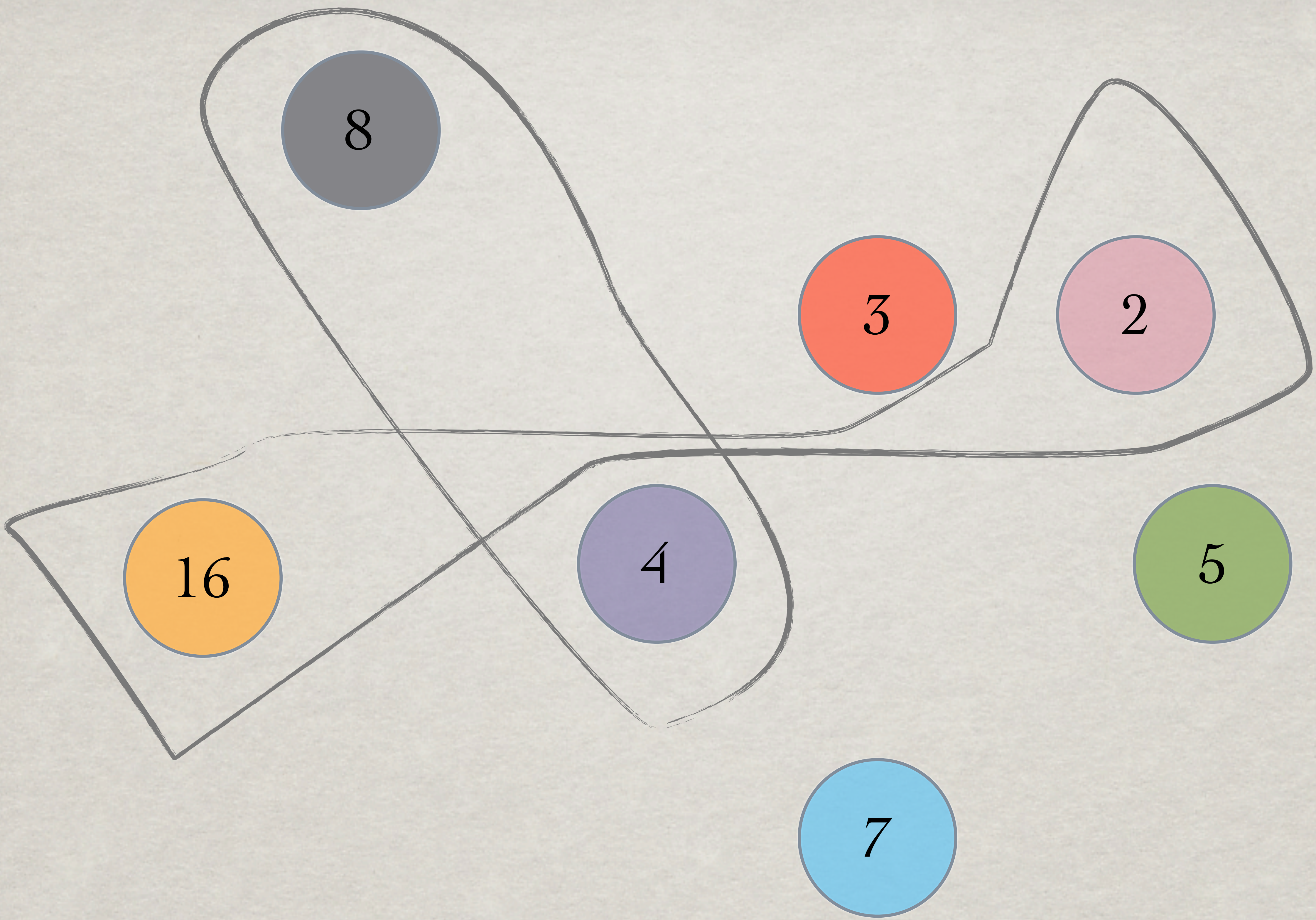
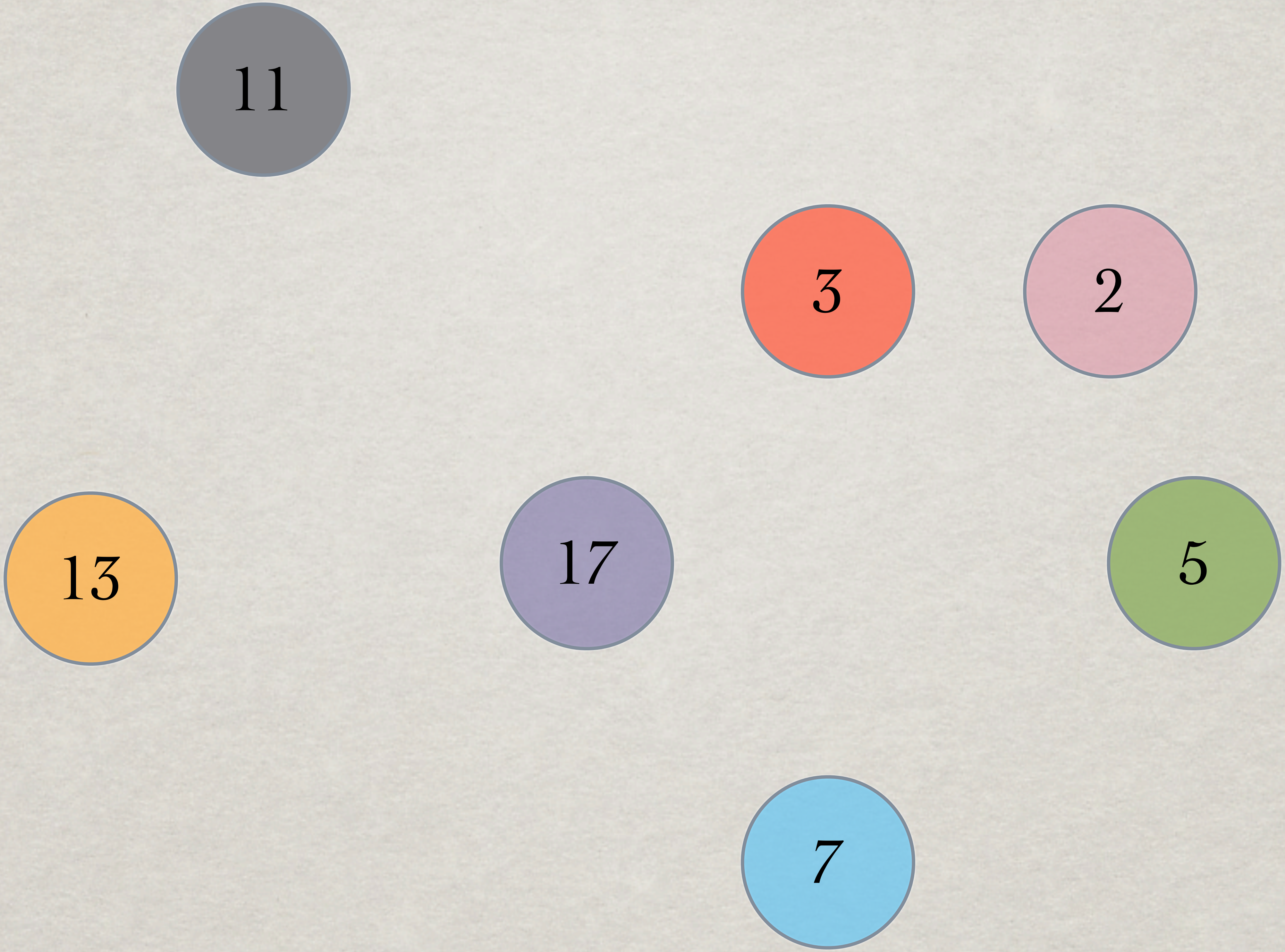
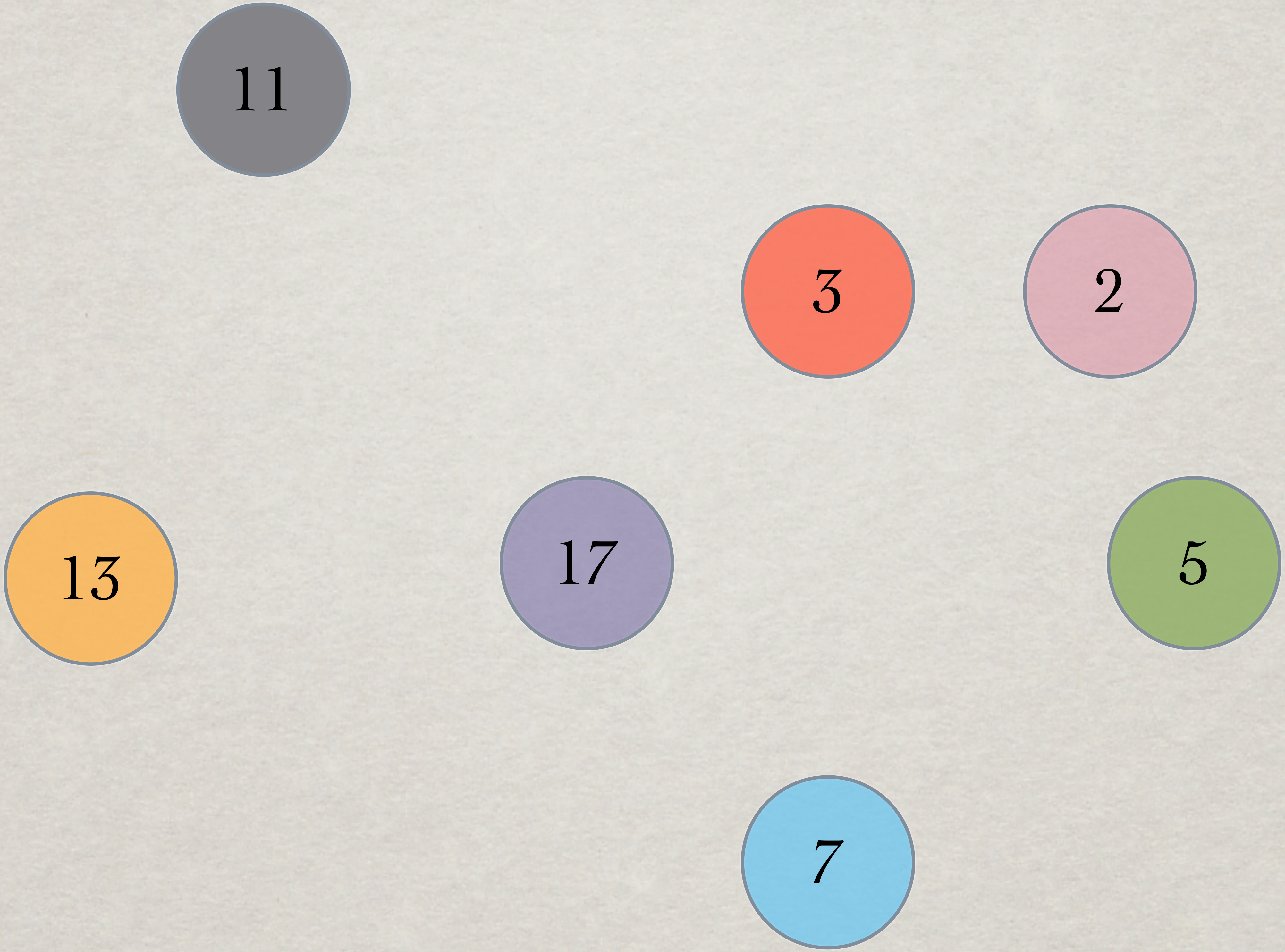
7

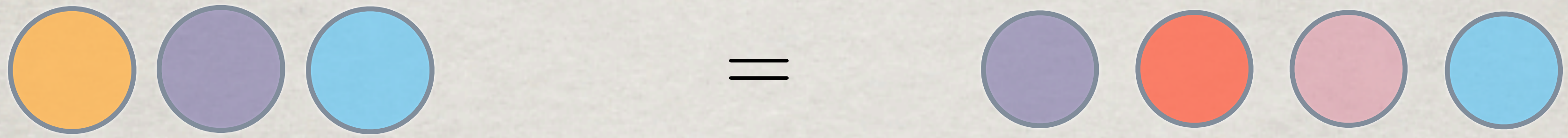

8
3
2
16
4
5
7

Maybe I needed to
chose a different set of numbers?

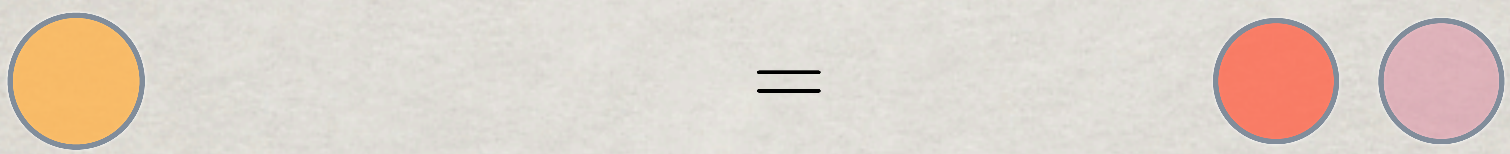

11
3
2
13
17
5
7

What's special about those numbers?


11
3
2
13
17
5
7



Common colors/numbers will cancel.



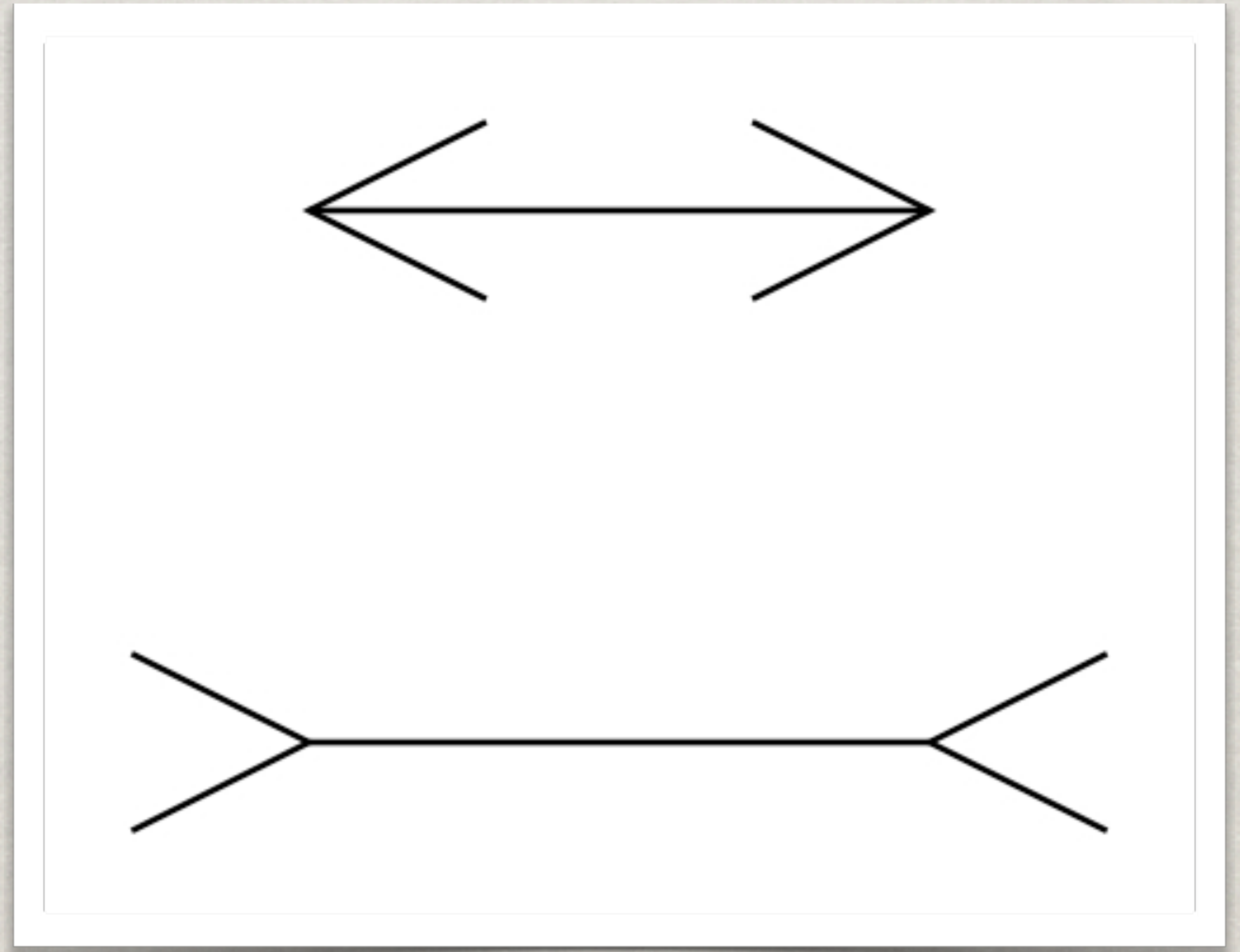
Not everything will cancel (why?)

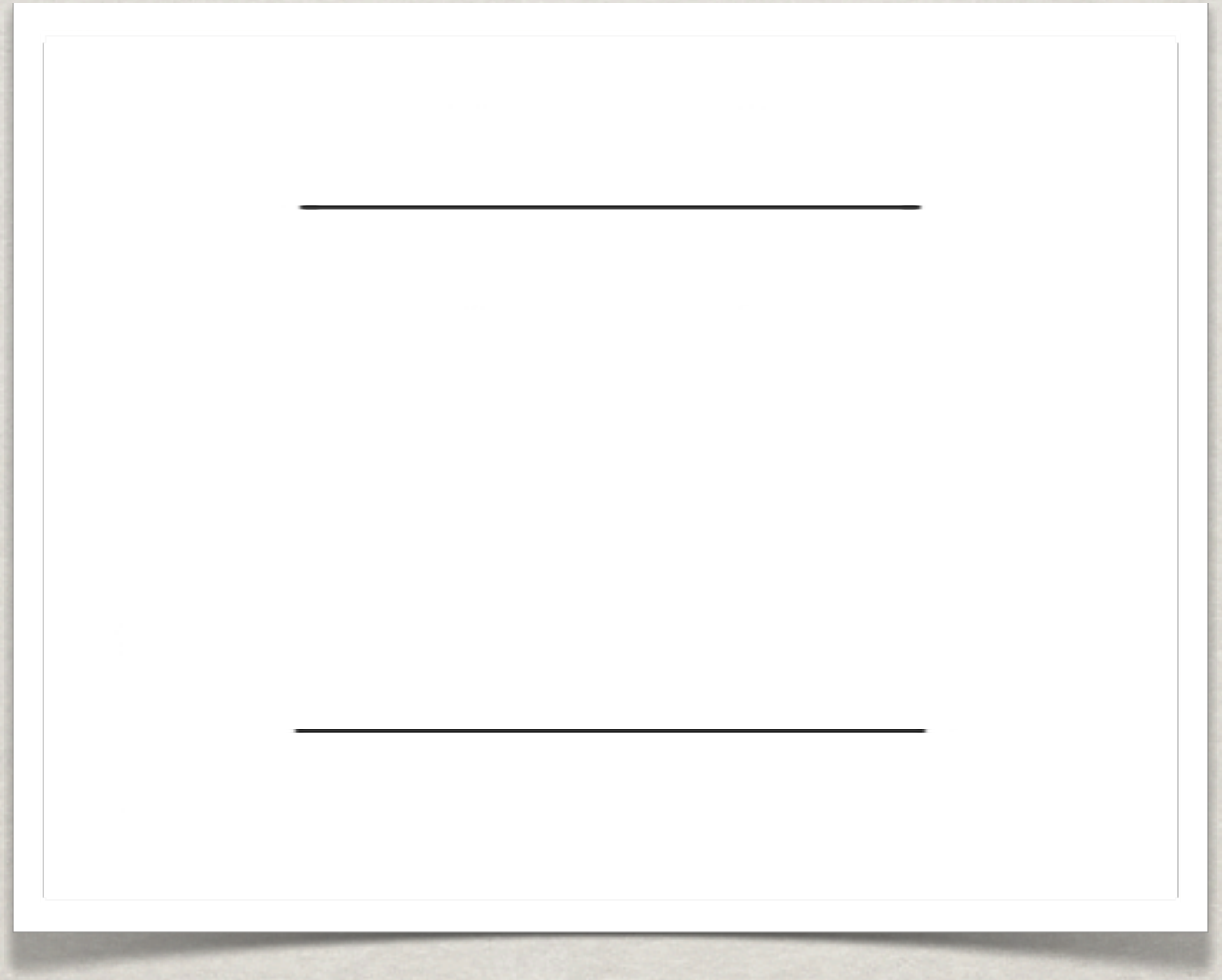
And what remains cannot be the same.
(why?)

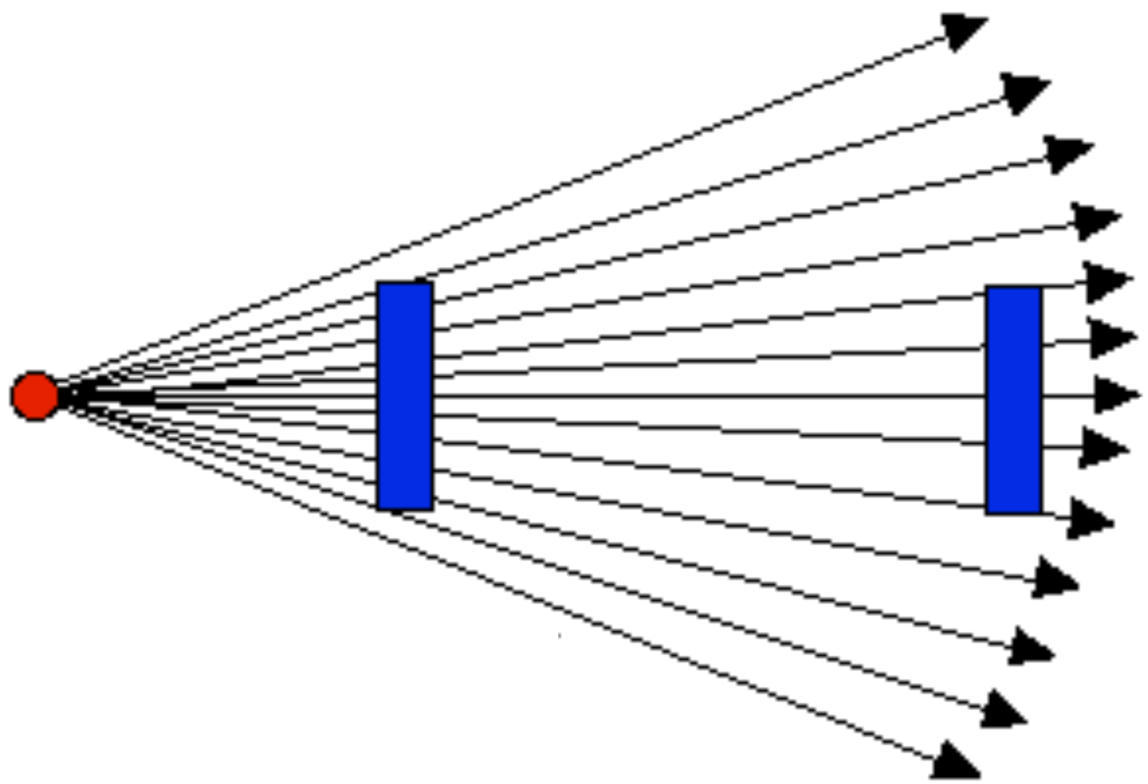
Some more

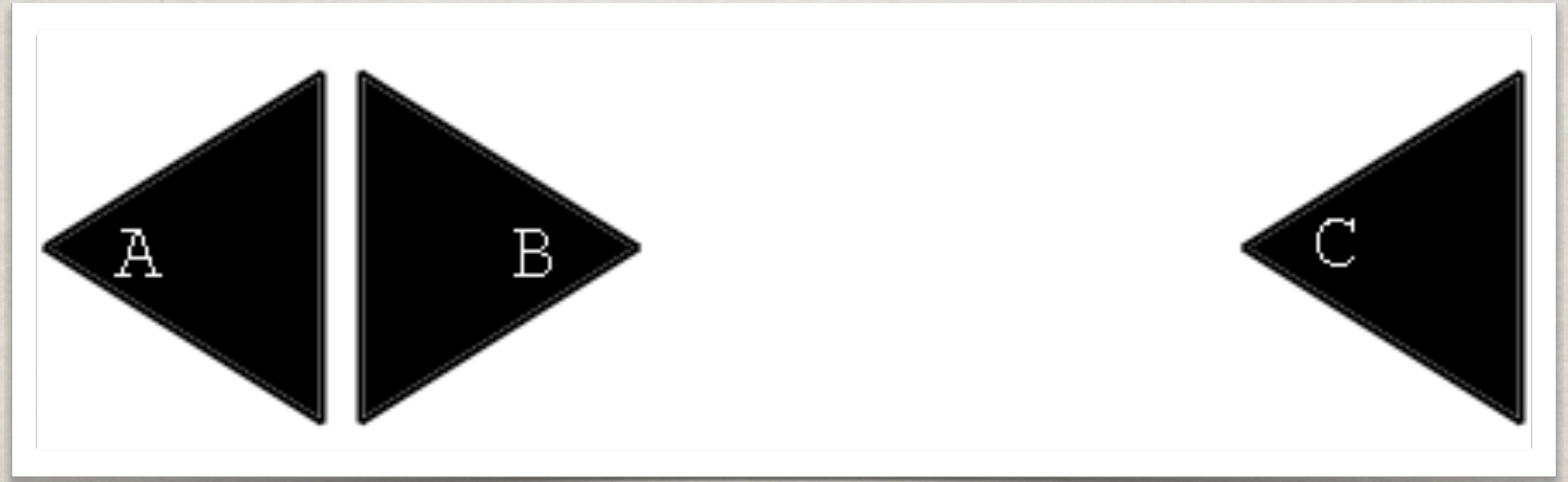
“which is bigger”

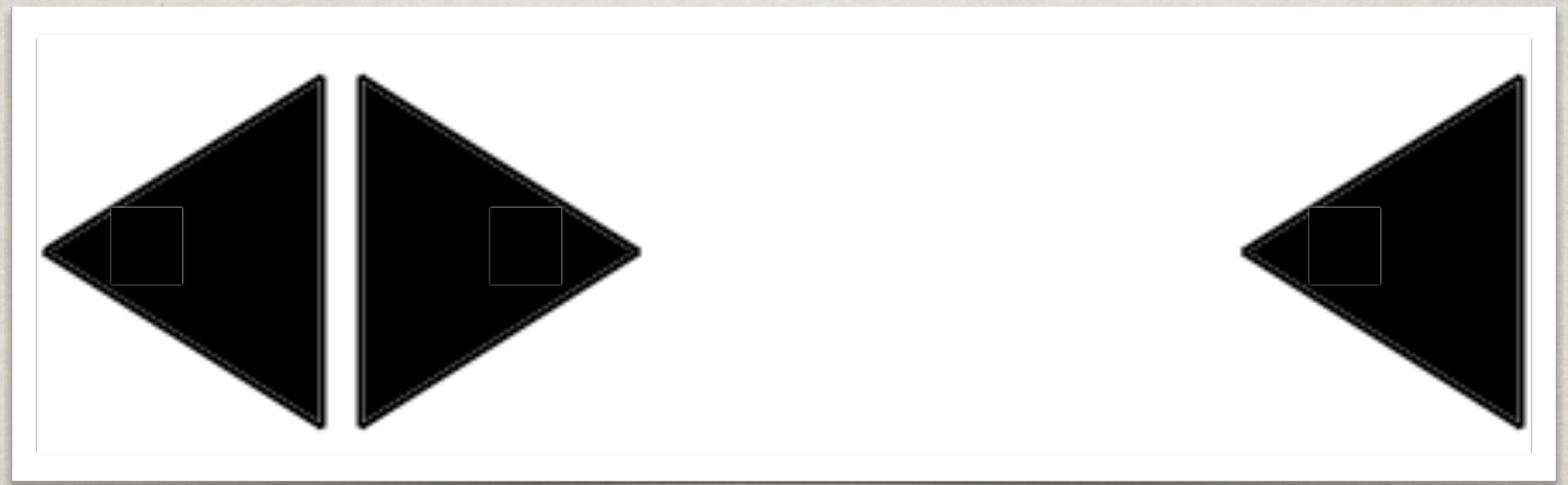
questions…

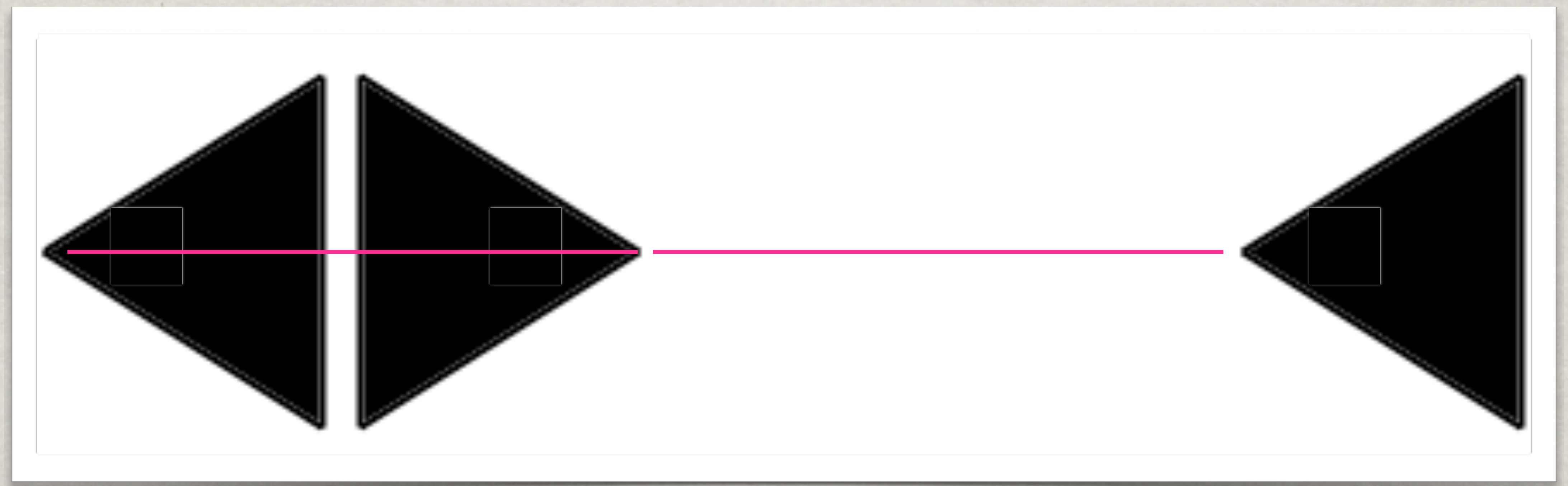







A
B
C





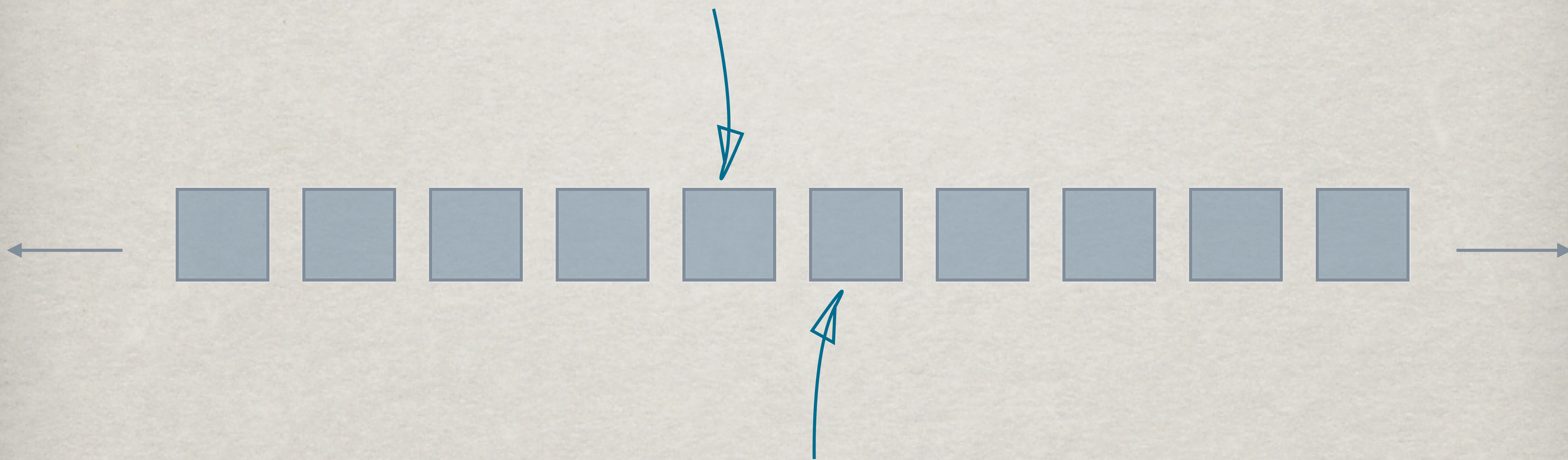
Story Time!

Once Upon a Time...

...there was a hotel with just one floor,
but infinitely many rooms.


1
2
3
4
5
6
7
8
9
10

Room# 103920320935893840928942303242398109



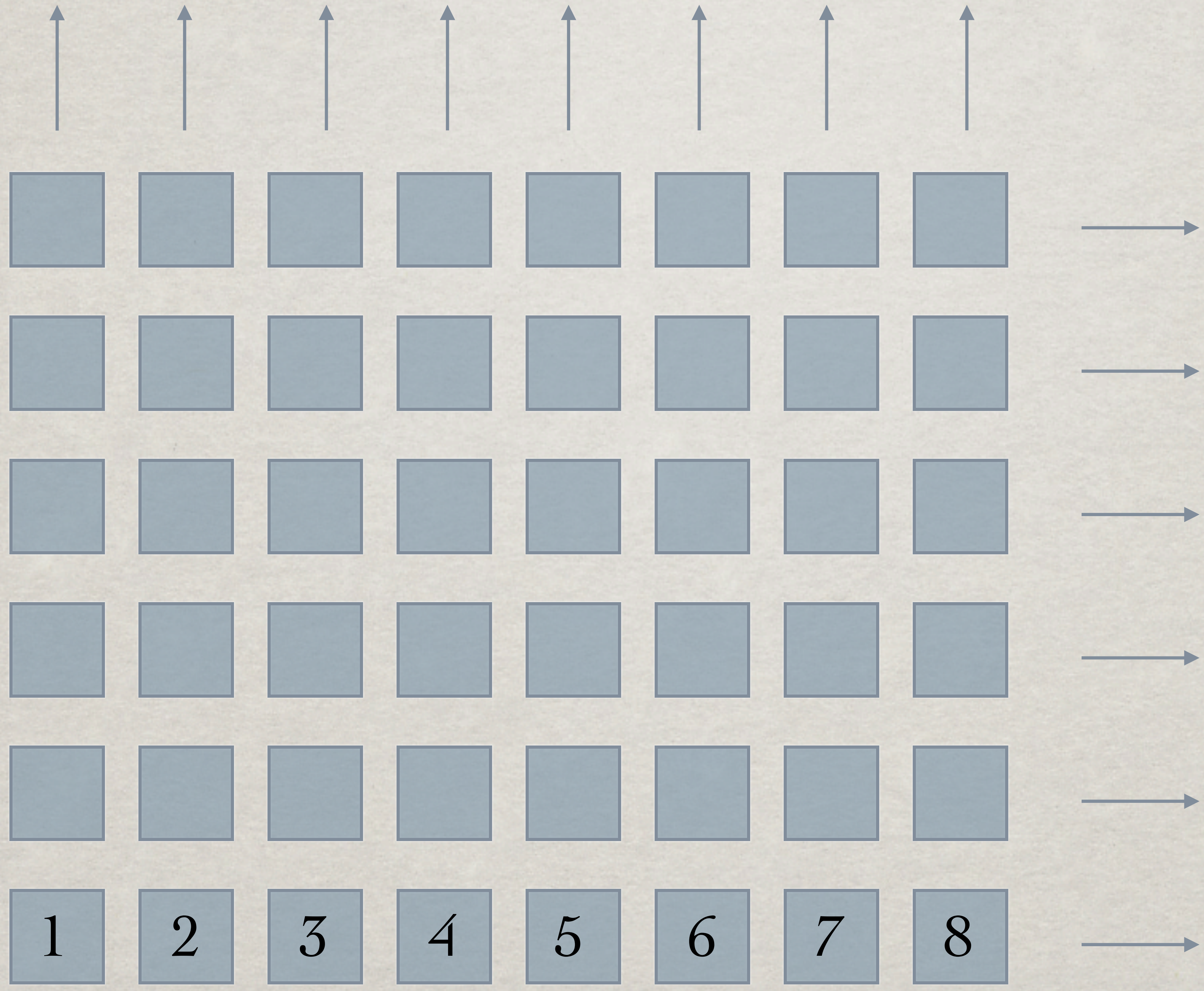
Room# 103920320935893840928942303242398111

One fine day...

there was an emergency in a hotel
from a neighboring town.

This hotel had infinitely many rooms…

...and infinitely many floors!


1
2
3
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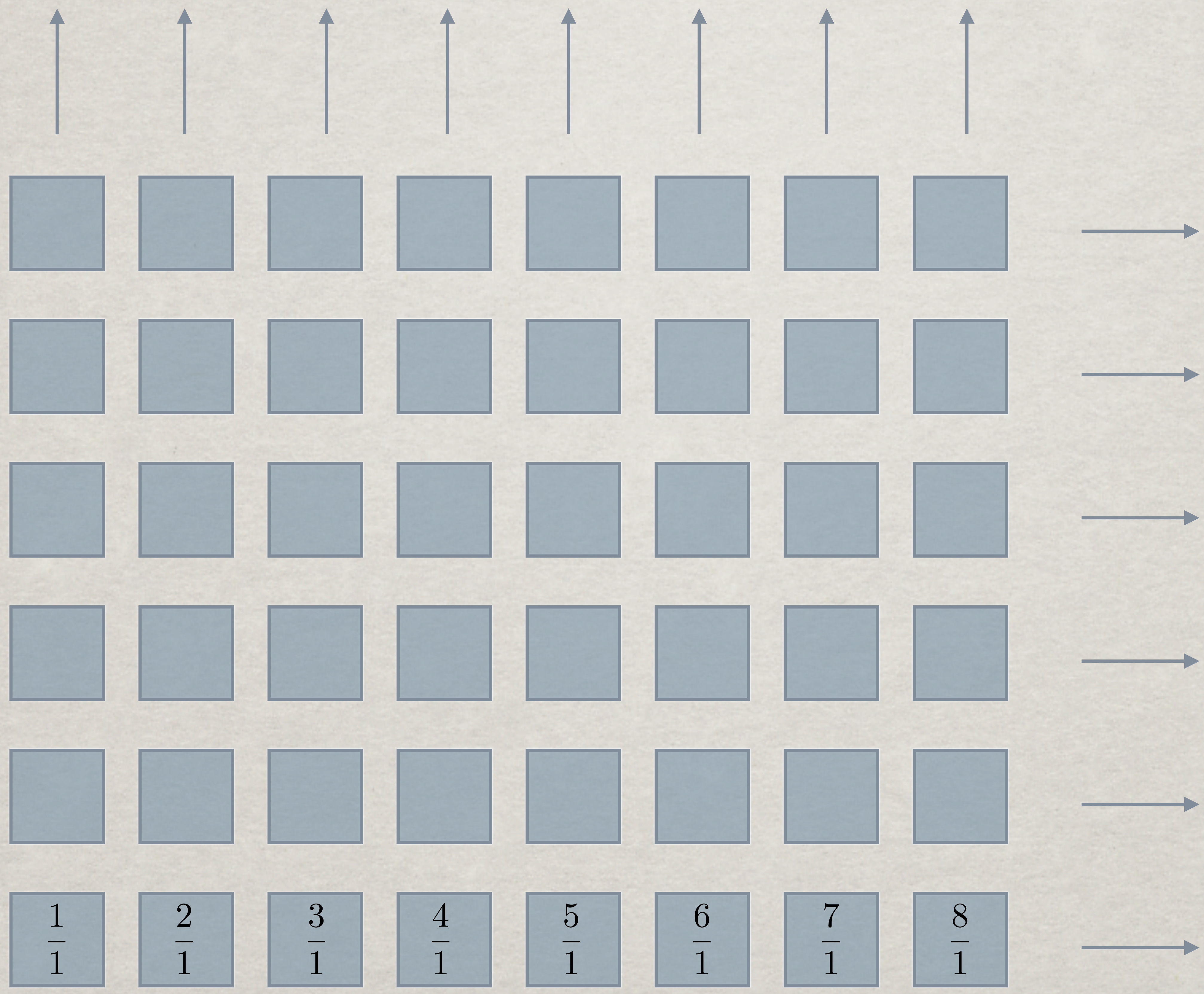
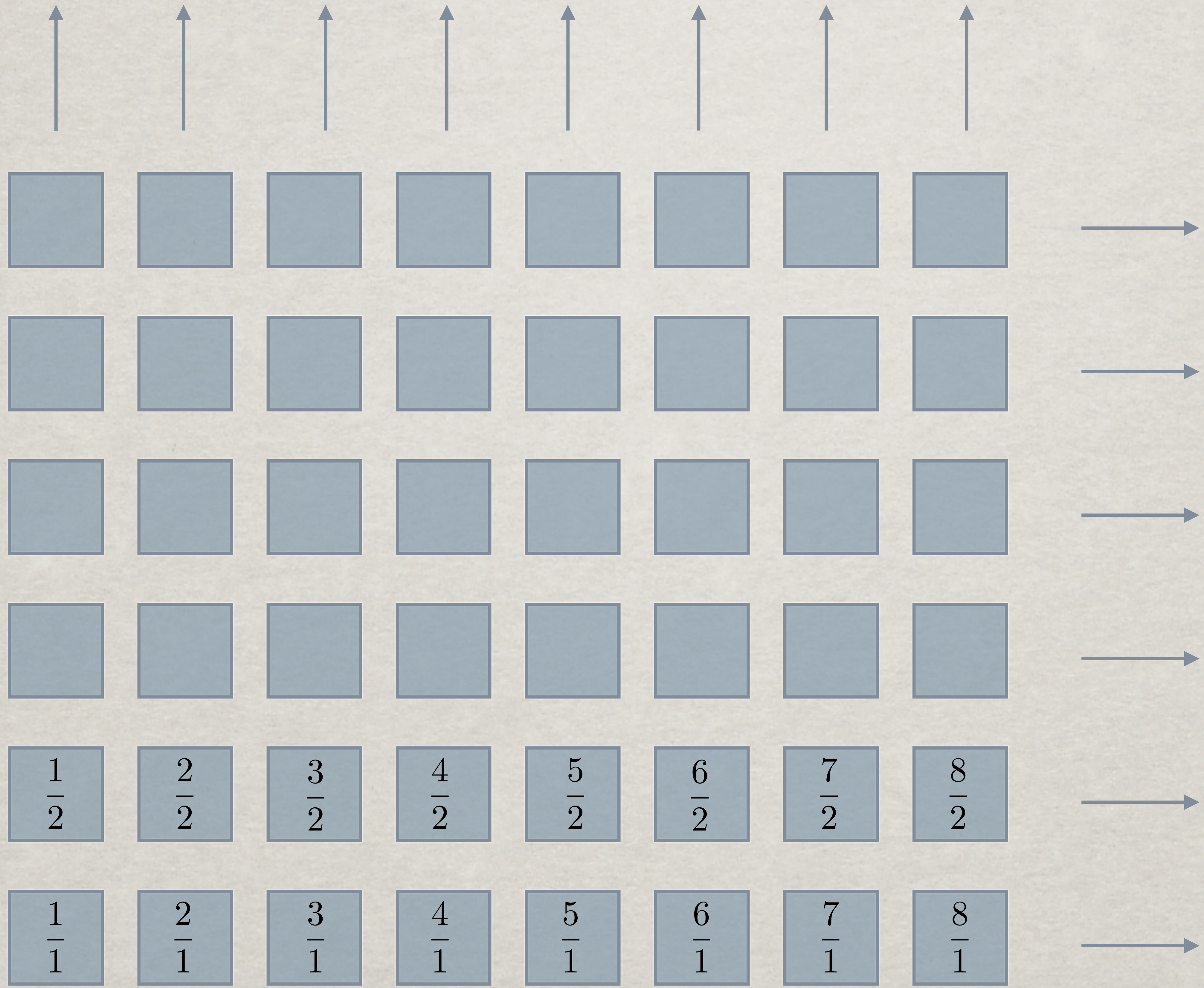
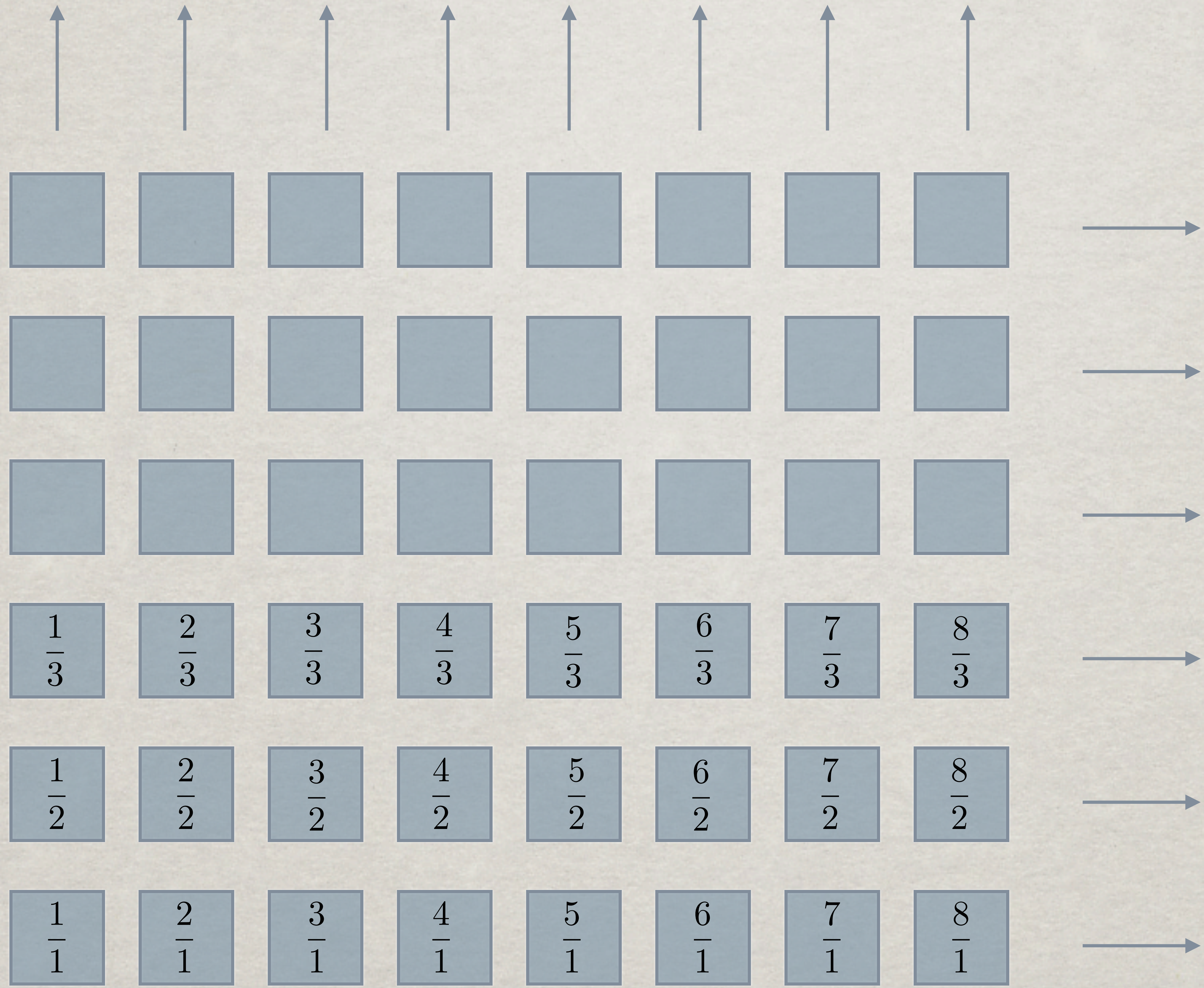
6
7
8



$\frac{1}{1}$
$\frac{2}{1}$
$\frac{3}{1}$
$\frac{4}{1}$
$\frac{5}{1}$
$\frac{6}{1}$
$\frac{7}{1}$
$\frac{8}{1}$

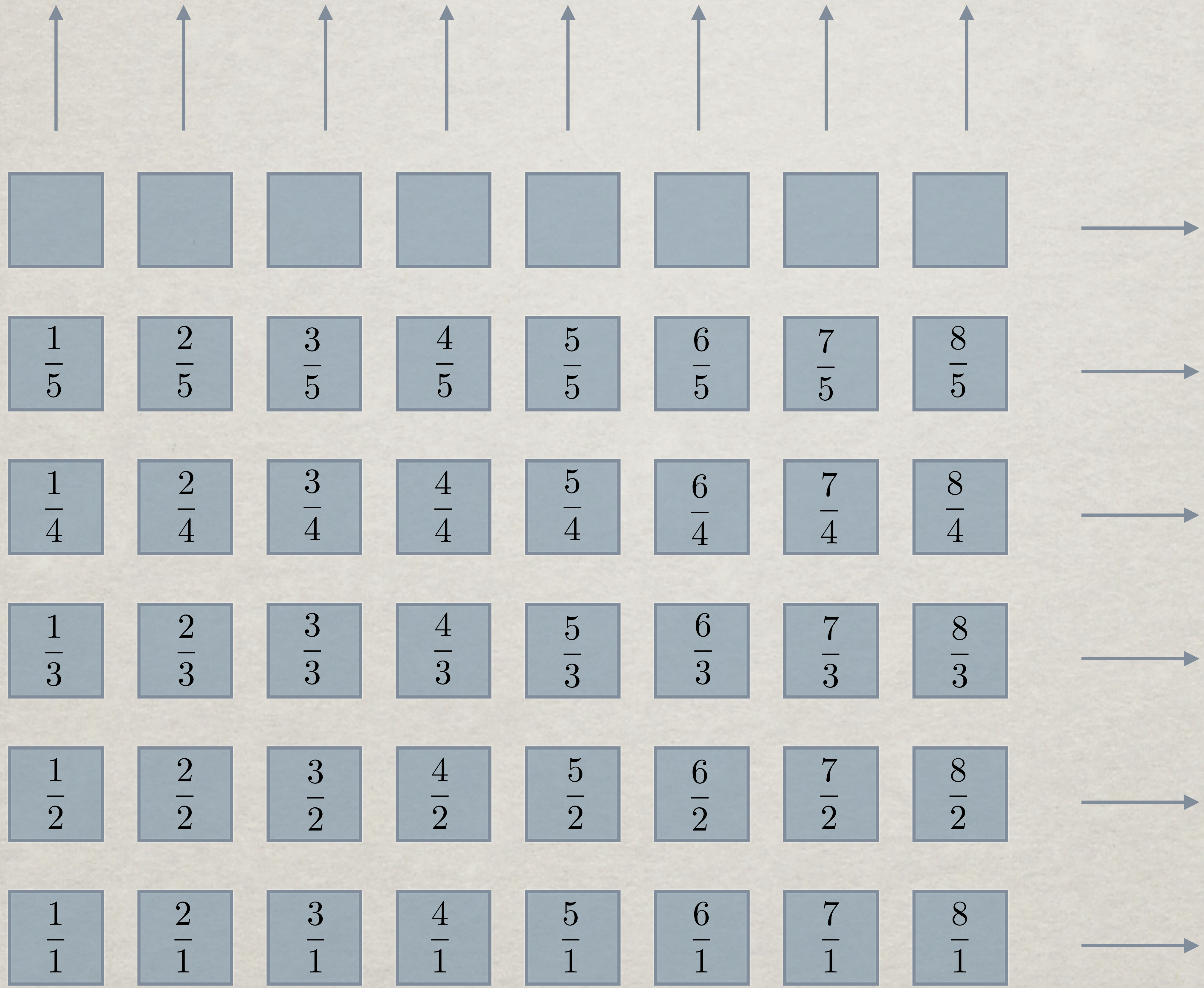


$\frac{1}{2}$ $\frac{2}{2}$ $\frac{3}{2}$ $\frac{4}{2}$ $\frac{5}{2}$ $\frac{6}{2}$ $\frac{7}{2}$ $\frac{8}{2}$
$\frac{1}{1}$ $\frac{2}{1}$ $\frac{3}{1}$ $\frac{4}{1}$ $\frac{5}{1}$ $\frac{6}{1}$ $\frac{7}{1}$ $\frac{8}{1}$

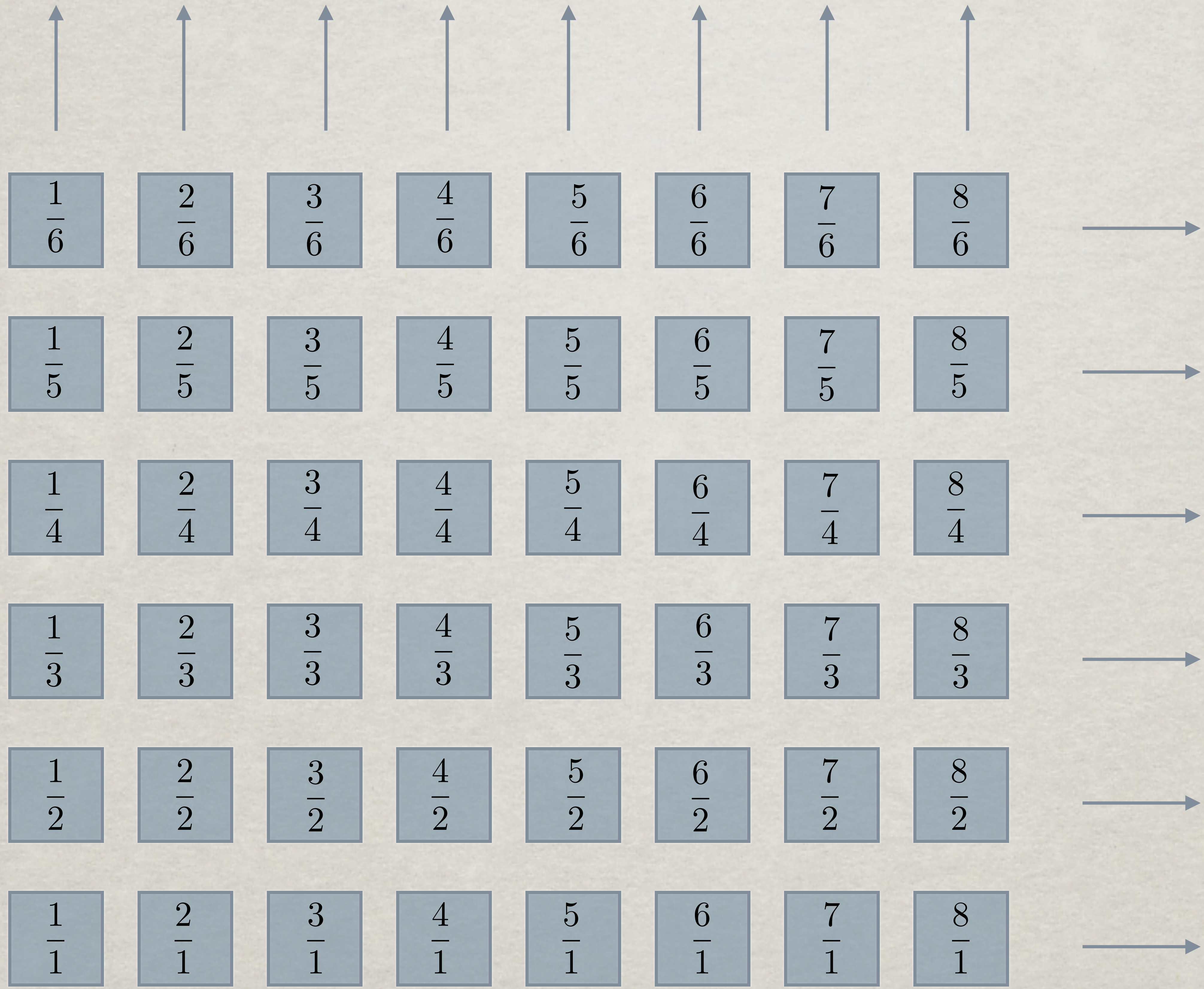


1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
1/2 2/2 3/2 4/2 5/2 6/2 7/2 8/2
1/1 2/1 3/1 4/1 5/1 6/1 7/1 8/1

$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$
$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$
$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$	$\frac{8}{1}$

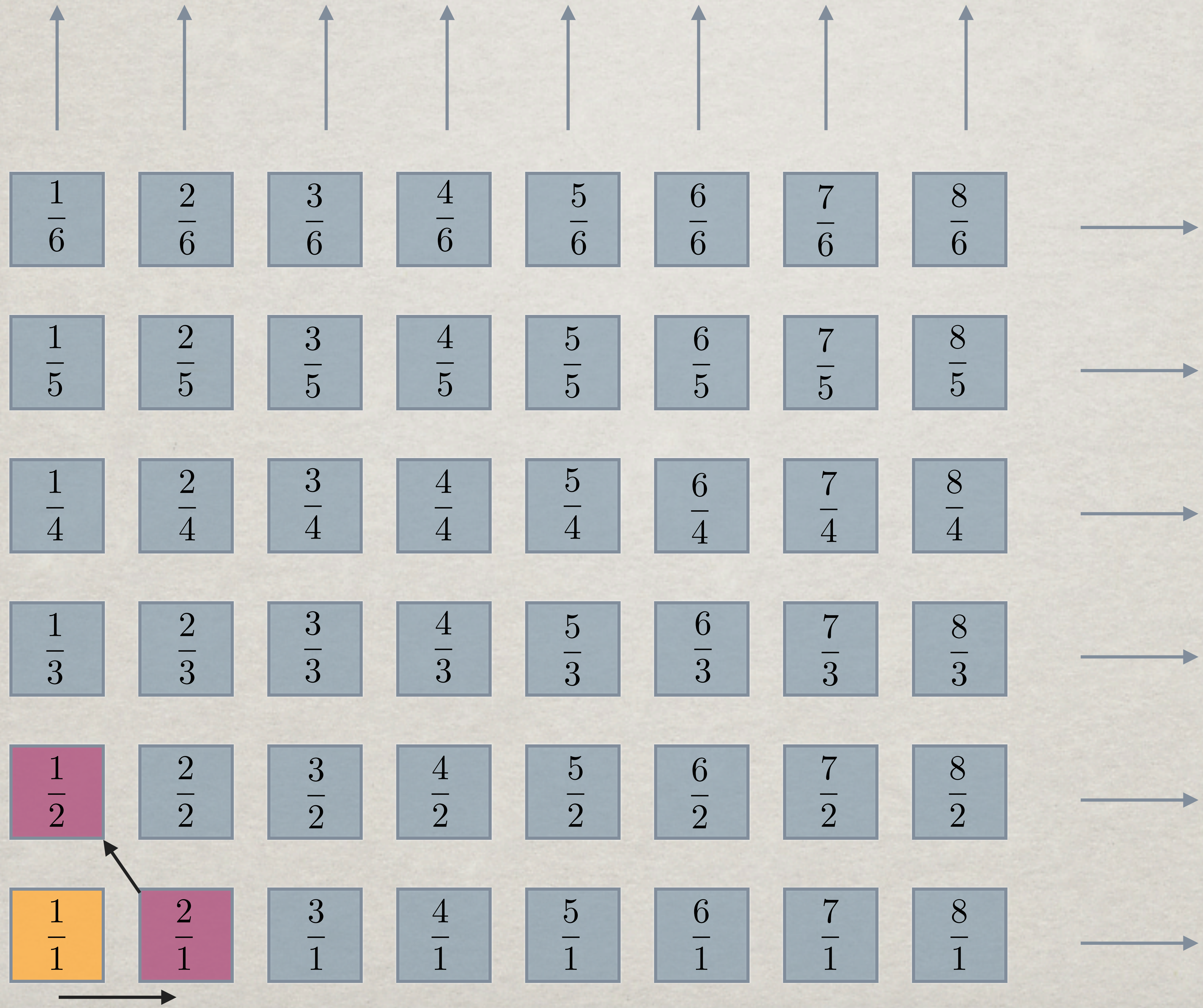


1/5 2/5 3/5 4/5 5/5 6/5 7/5 8/5
1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
1/2 2/2 3/2 4/2 5/2 6/2 7/2 8/2
1/1 2/1 3/1 4/1 5/1 6/1 7/1 8/1

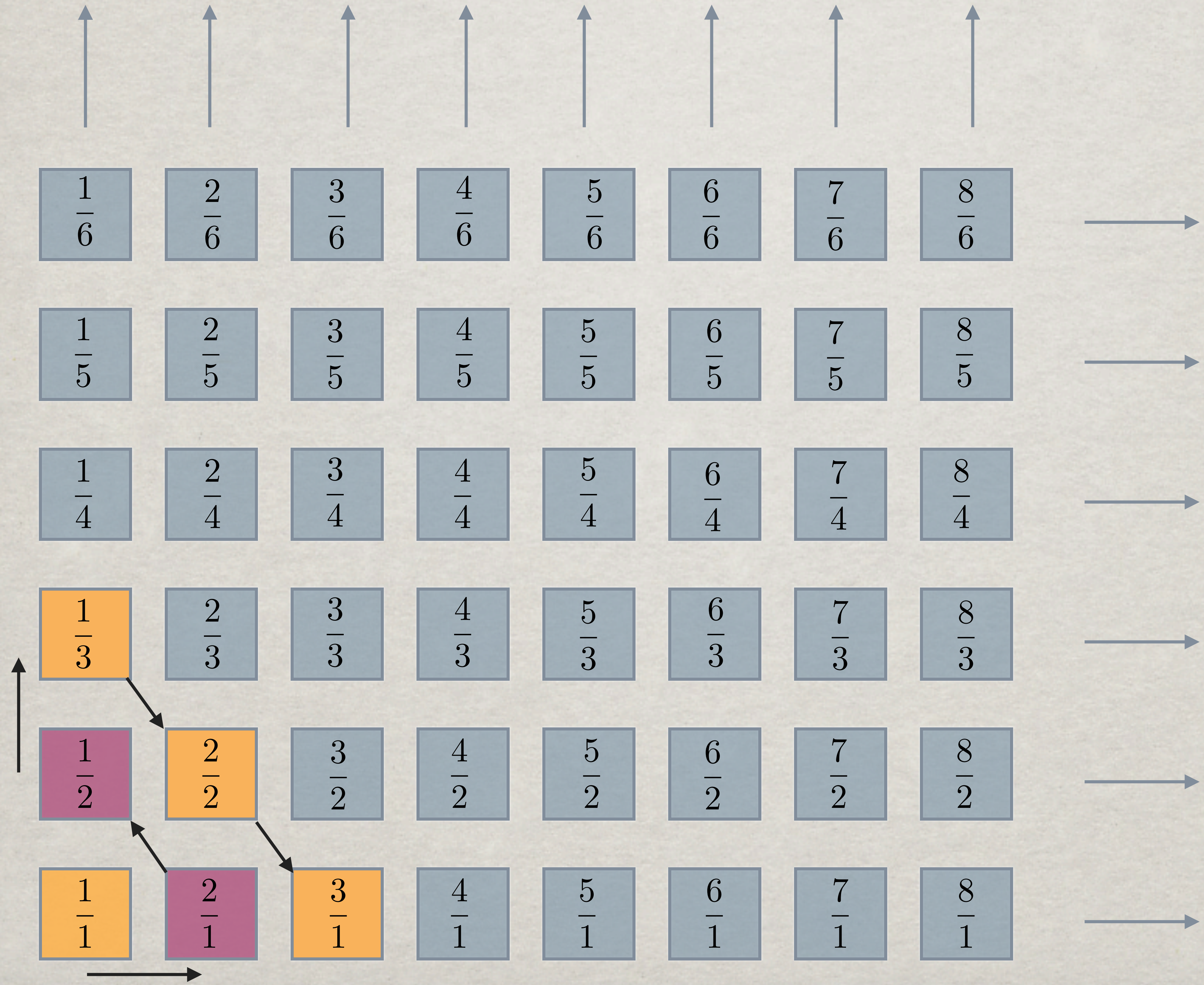

1/6 2/6 3/6 4/6 5/6 6/6 7/6 8/6
1/5 2/5 3/5 4/5 5/5 6/5 7/5 8/5
1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
1/2 2/2 3/2 4/2 5/2 6/2 7/2 8/2
1/1 2/1 3/1 4/1 5/1 6/1 7/1 8/1

So, how do we shift all these people into the “smaller” hotel?

$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$
$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$
$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$
$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$	$\frac{8}{1}$



1/6 2/6 3/6 4/6 5/6 6/6 7/6 8/6
1/5 2/5 3/5 4/5 5/5 6/5 7/5 8/5
1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
1/2 2/2 3/2 4/2 5/2 6/2 7/2 8/2
1/1 2/1 3/1 4/1 5/1 6/1 7/1 8/1



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1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
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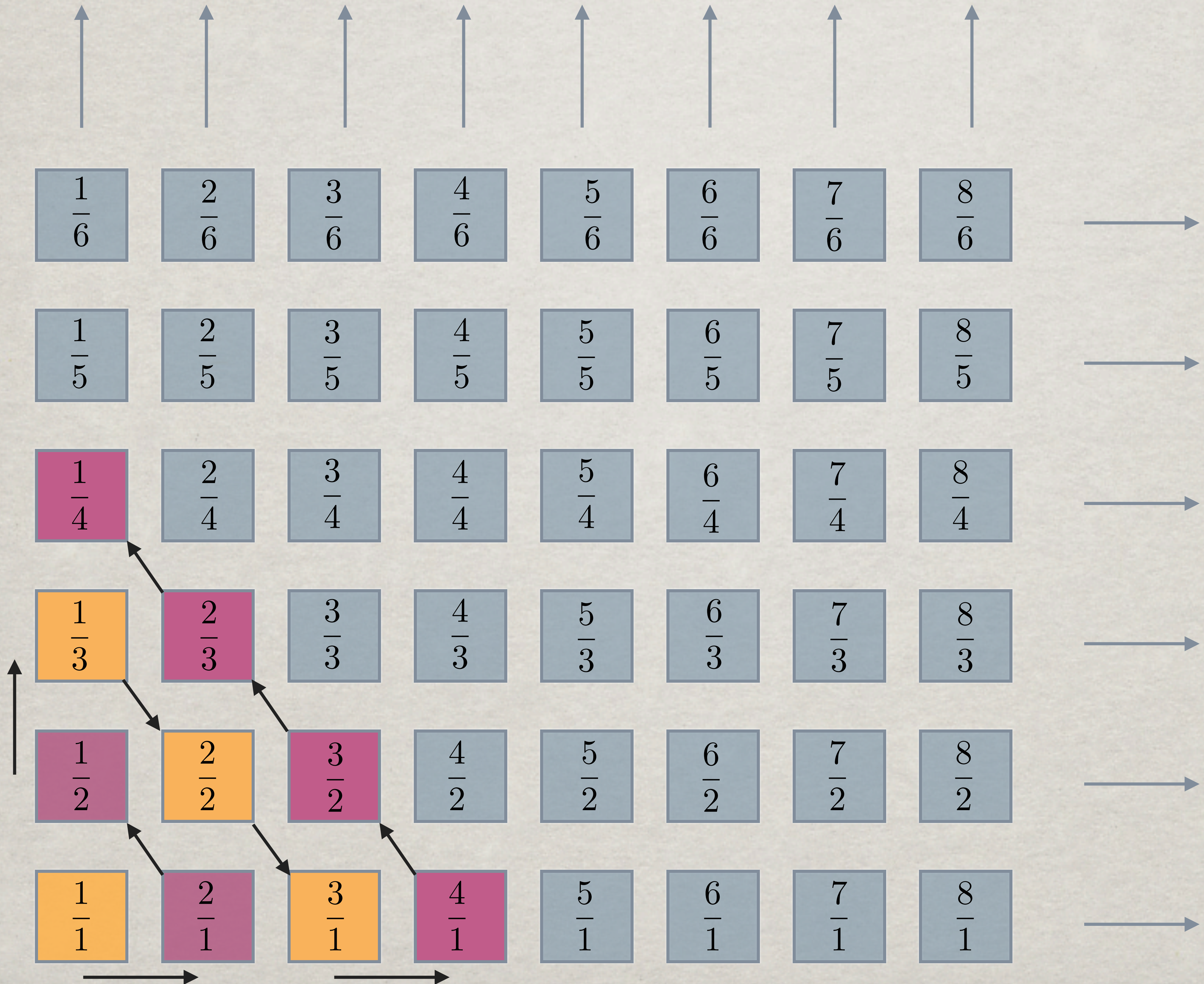

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1/5 2/5 3/5 4/5 5/5 6/5 7/5 8/5
1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
1/2 2/2 3/2 4/2 5/2 6/2 7/2 8/2
1/1 2/1 3/1 4/1 5/1 6/1 7/1 8/1

1/6 2/6 3/6 4/6 5/6 6/6 7/6 8/6
1/5 2/5 3/5 4/5 5/5 6/5 7/5 8/5
1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
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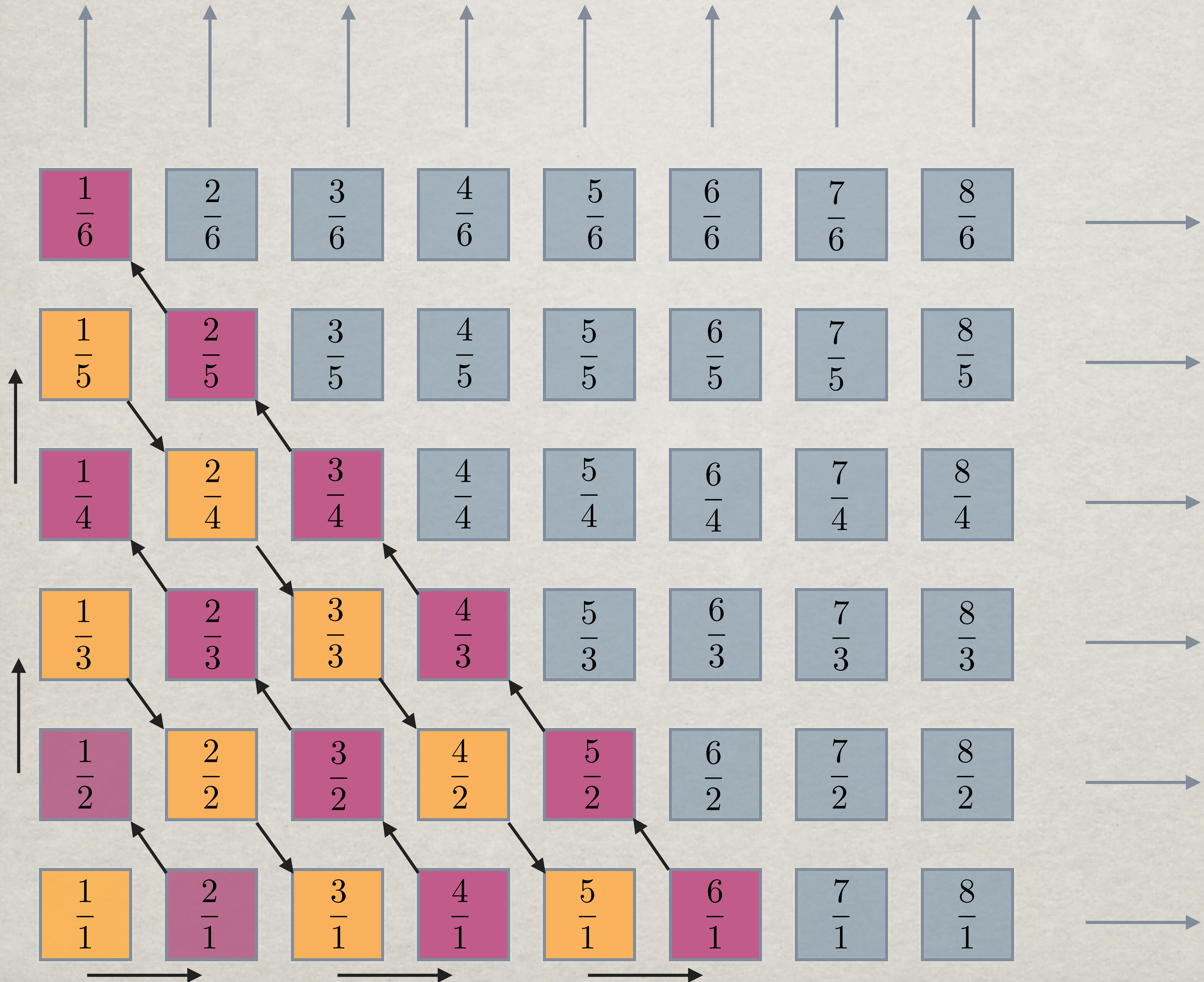

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1/5 2/5 3/5 4/5 5/5 6/5 7/5 8/5
1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
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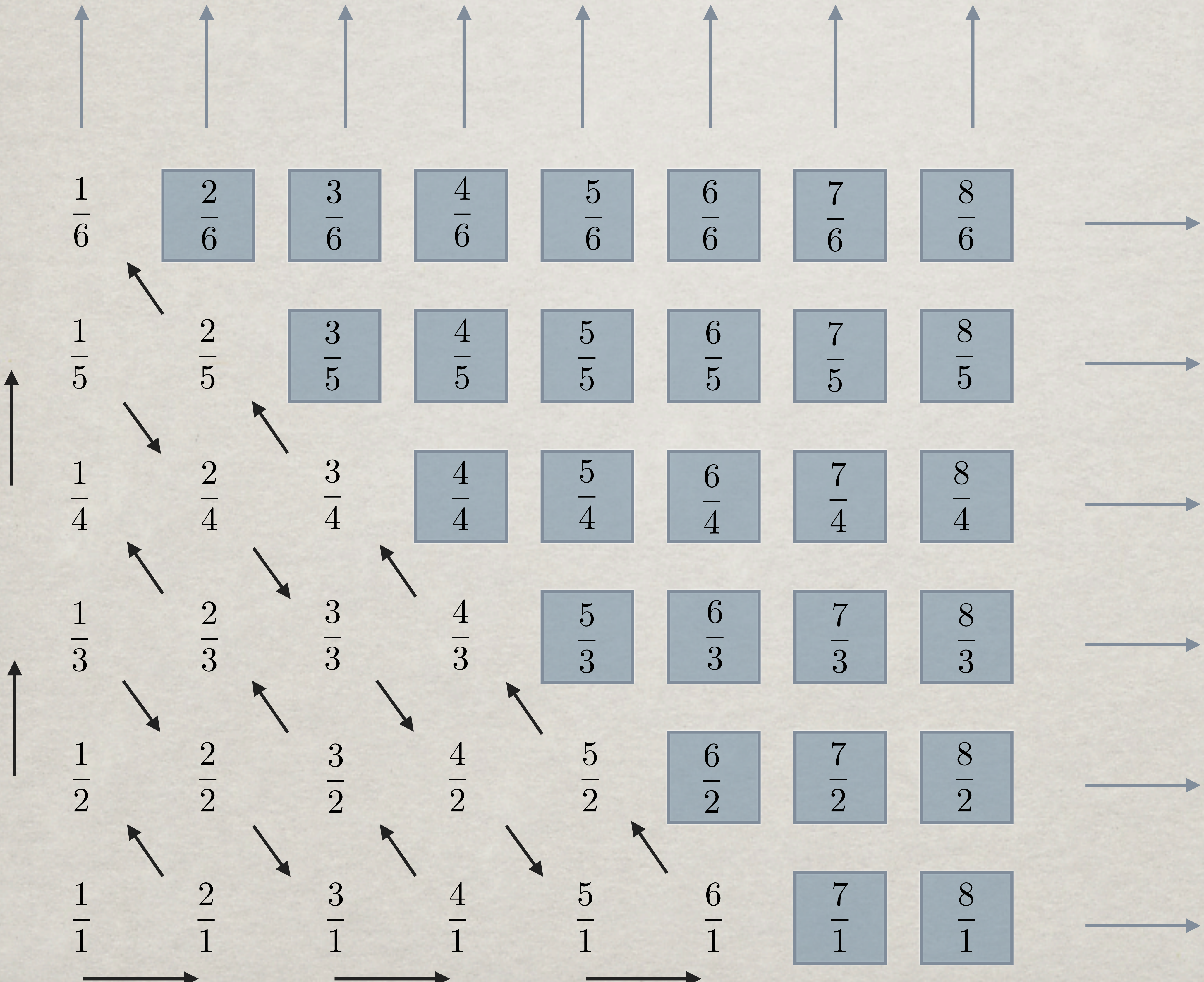
1/1 2/1 3/1 4/1 5/1 6/1 7/1 8/1



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1/5 2/5 3/5 4/5 5/5 6/5 7/5 8/5
1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
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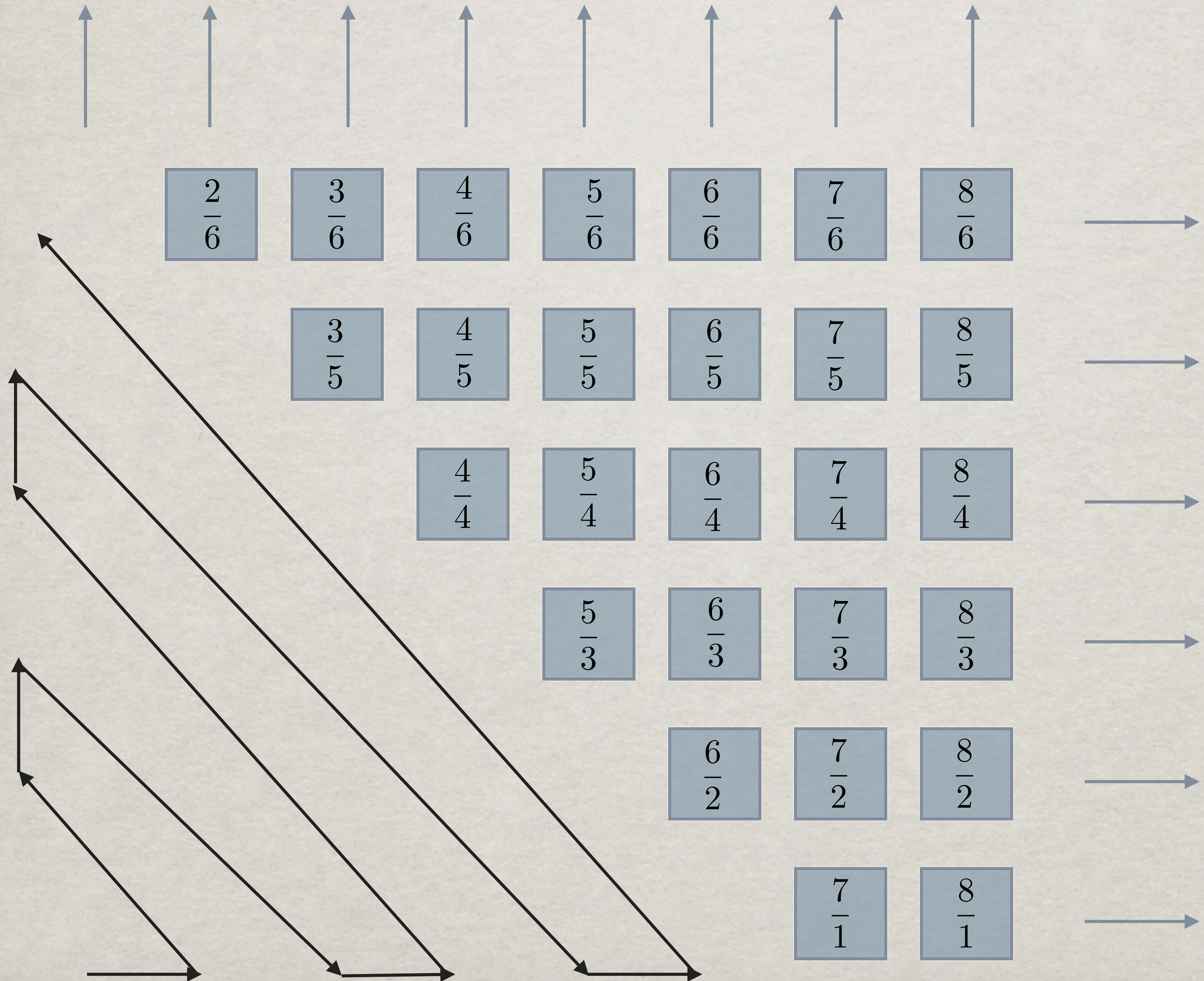
1/1 2/1 3/1 4/1 5/1 6/1 7/1 8/1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3/5 4/5 5/5 6/5 7/5 8/5
4/4 5/4 6/4 7/4 8/4
5/3 6/3 7/3 8/3
6/2 7/2 8/2
7/1 8/1

1/6 2/6 3/6 4/6 5/6 6/6 7/6 8/6
1/5 2/5 3/5 4/5 5/5 6/5 7/5 8/5
1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
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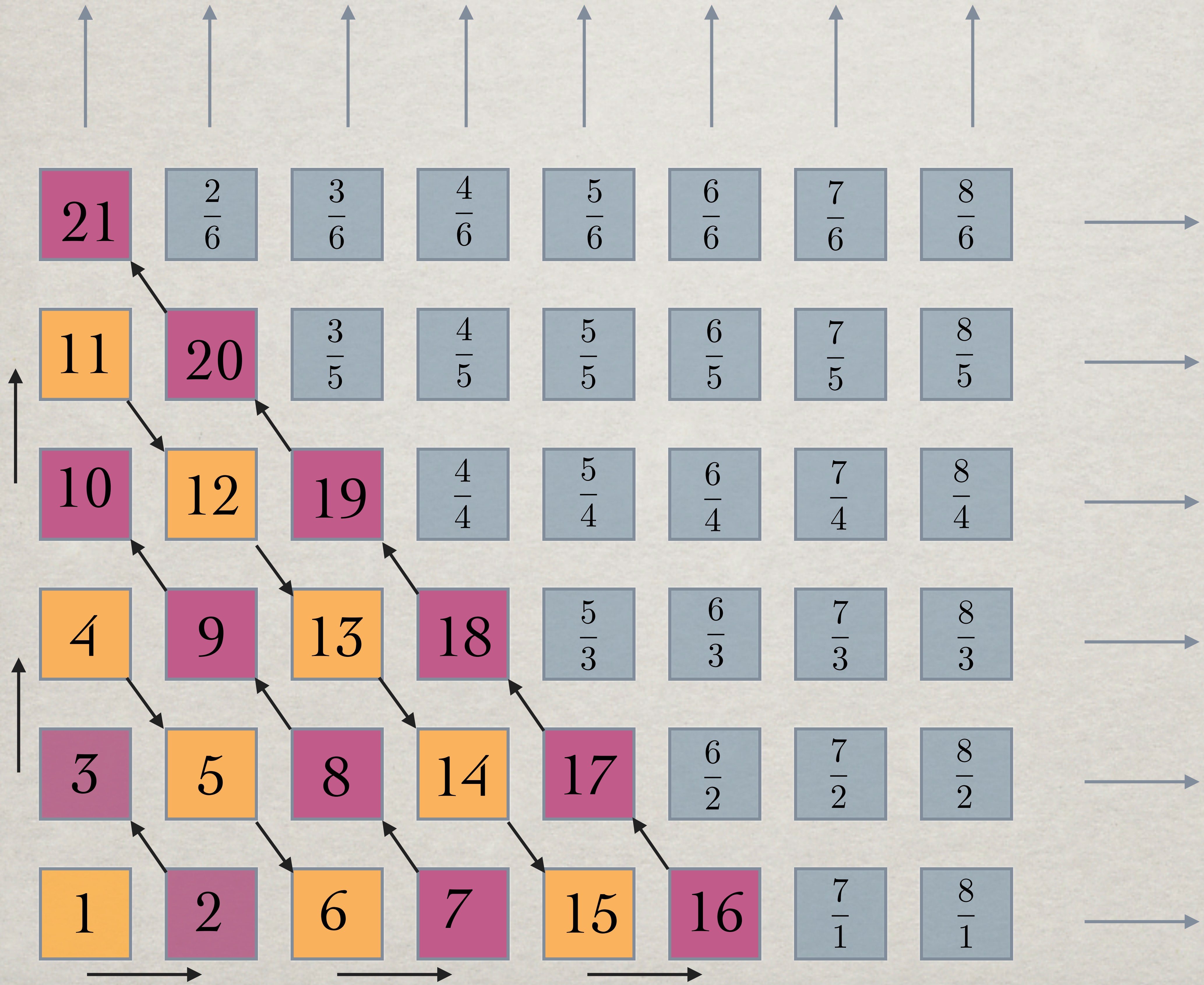


21 2/6 3/6 4/6 5/6 6/6 7/6 8/6
11 20 3/5 4/5 5/5 6/5 7/5 8/5
10 12 19 4/4 5/4 6/4 7/4 8/4
4 9 13 18 5/3 6/3 7/3 8/3
3 5 8 14 17 6/2 7/2 8/2
1 2 6 7 15 16 7/1 8/1

Notice that we have effectively placed
the rooms of the “bigger” hotel
in a *one-one correspondence*
with the rooms of the “smaller” hotel.

So the smaller hotel wasn’t smaller after all!

But remember that it is an emergency,
and it’s not a good idea to keep people waiting to
get out of their rooms.

So for every room number of the form

$$\frac{p}{q}$$

can we generate an **unique** room number in the new hotel?

That is, we don't want people in different rooms

$$\frac{p}{q}, \frac{a}{b}$$

$p \neq a$ or

$q \neq b$

to end up having to share the same room in the new hotel.

That is, we don't want people in different rooms

$$\frac{p}{q}, \frac{a}{b}$$

$p \neq a$ or

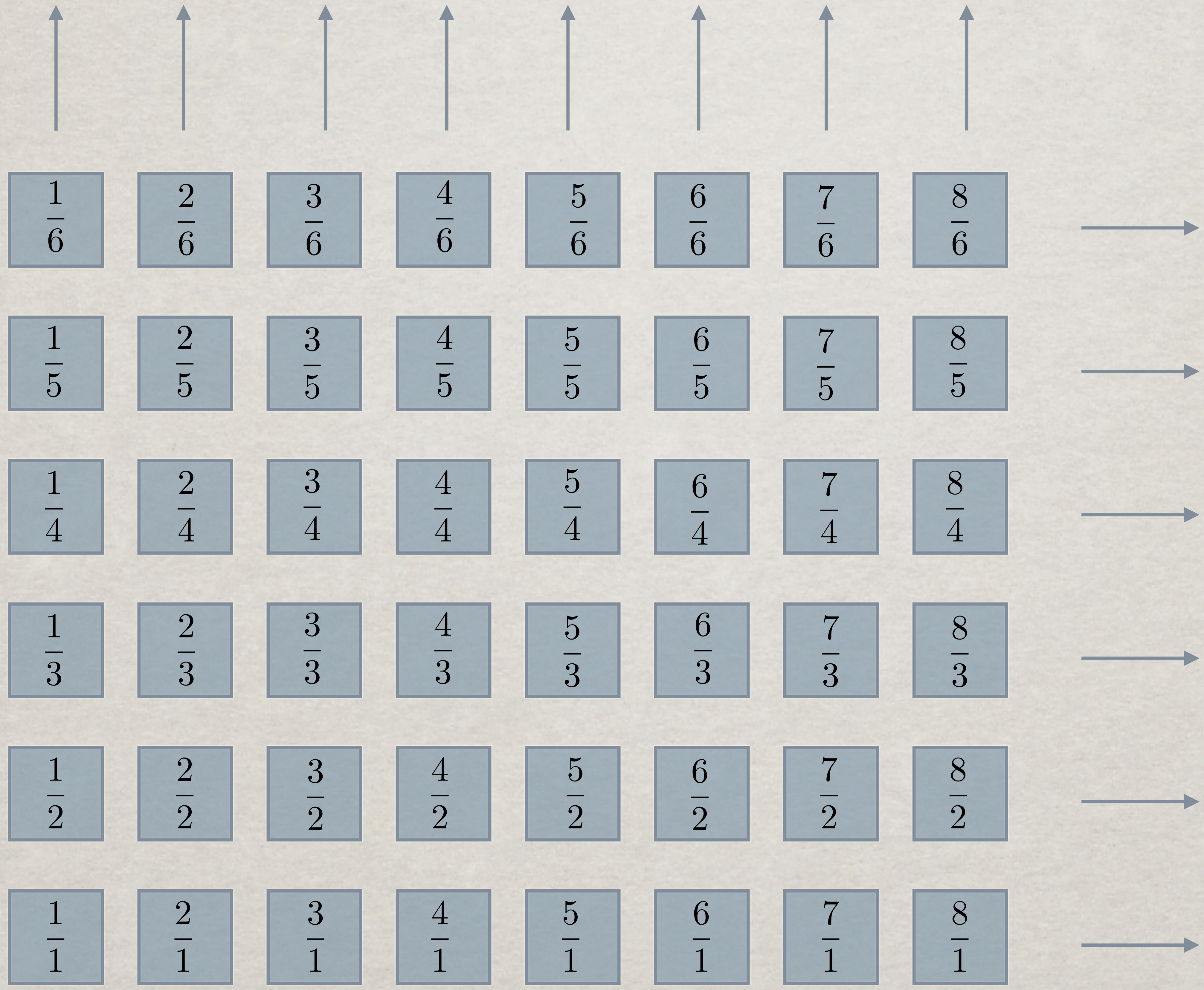
$q \neq b$

to end up having to share the same room in the new hotel.

(This is the same as looking for a one-to-one correspondence.)

Suggestions?

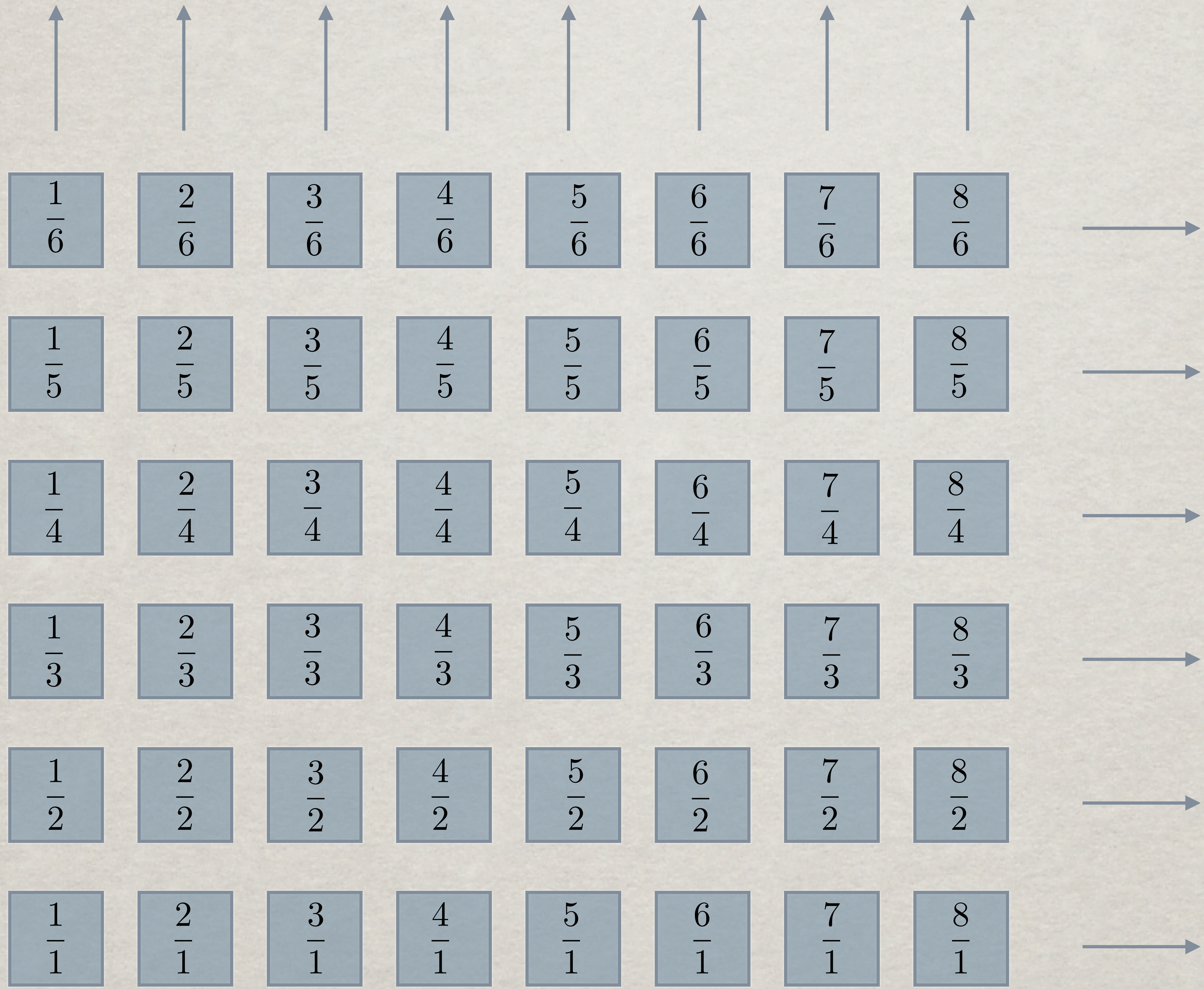
$$\frac{p}{q} \longrightarrow (p+q)$$


1/6 2/6 3/6 4/6 5/6 6/6 7/6 8/6
1/5 2/5 3/5 4/5 5/5 6/5 7/5 8/5
1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
1/2 2/2 3/2 4/2 5/2 6/2 7/2 8/2
1/1 2/1 3/1 4/1 5/1 6/1 7/1 8/1

7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9

7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9

$$\frac{p}{q} \longrightarrow (p \cdot q)$$


1/6 2/6 3/6 4/6 5/6 6/6 7/6 8/6
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1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
1/3 2/3 3/3 4/3 5/3 6/3 7/3 8/3
1/2 2/2 3/2 4/2 5/2 6/2 7/2 8/2
1/1 2/1 3/1 4/1 5/1 6/1 7/1 8/1

6	12	18	24	30	36	42	48
5	10	15	20	25	30	35	40
4	8	12	16	20	24	28	32
3	6	9	12	15	18	21	24
2	4	6	8	10	12	14	16
1	2	3	4	5	6	7	8

6	12	18	24	30	36	42	48
5	10	15	20	25	30	35	40
4	8	12	16	20	24	28	32
3	6	9	12	15	18	21	24
2	4	6	8	10	12	14	16
1	2	3	4	5	6	7	8

$$\frac{p}{q} \rightarrow (2^p 3^q)$$

$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$
$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$
$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$
$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$	$\frac{8}{1}$

162	324	648	1296	2592	5184	10368	20736
54	108	216	432	864	1728	3456	6912
18	36	72	144	288	576	1152	2304
6	12	24	48	96	192	384	768

No repeats so far...
and no repeats forever (why?)

Consider → $\frac{p}{q}, \frac{a}{b}$

$p \neq a$ or

$q \neq b$

Consider → $\frac{p}{q}, \frac{a}{b}$ $p \neq a$ or $q \neq b$

$$2^p 3^q = 2^a 3^b$$ ← Assume

Consider $\rightarrow$ $$\frac{p}{q}, \frac{a}{b}$$ $p \neq a$ or $q \neq b$

$$2^p 3^q = 2^a 3^b$$ $\leftarrow$ Assume

$$2^{(p-a)} 3^{(q-b)} = 1$$

Consider → $\frac{p}{q}, \frac{a}{b}$ $\quad p \neq a$ or $q \neq b$

$$2^p 3^q = 2^a 3^b$$ ← Assume

$$2^{(p-a)} 3^{(q-b)} = 1$$

$$p - a = 0 \text{ and } q - b = 0$$

Consider → $\frac{p}{q}, \frac{a}{b}$ $p \neq a$ or $q \neq b$

$$2^p 3^q = 2^a 3^b$$ ← Assume

$$2^{(p-a)} 3^{(q-b)} = 1$$

$$p - a = 0 \text{ and } q - b = 0$$

$$p = a \text{ and } q = b$$

Consider → $\frac{p}{q}, \frac{a}{b}$

$p \neq a$ or $q \neq b$

$$2^p 3^q = 2^a 3^b$$ ← Assume

$$2^{(p-a)} 3^{(q-b)} = 1$$

$$p - a = 0 \text{ and } q - b = 0$$

$$p = a \text{ and } q = b$$

↑ Contradiction!

Why did that work?
What about...?

$$\frac{p}{q} \rightarrow (2^p 4^q)$$

Any guesses?

$$\frac{p}{q} \to (2^p 4^q)$$

$$\frac{2}{3} \to (2^2 \cdot 4^3)$$

$$\frac{4}{2} \to (2^4 \cdot 4^2)$$

$$\frac{p}{q} \to (2^p 4^q)$$

$$\frac{2}{3} \to (2^2 \cdot 4^3)$$

$$\frac{4}{2} \to (2^4 \cdot 4^2)$$

256

256

So that failed.

Exercise: Figure out where the previous proof breaks down if you try to mimic it.

Consider → $\frac{p}{q}, \frac{a}{b}$

$p \neq a$ or $q \neq b$

$$2^p 3^q = 2^a 3^b$$ ← Assume

$$2^{(p-a)} 3^{(q-b)} = 1$$

$$p - a = 0 \text{ and } q - b = 0$$

$$p = a \text{ and } q = b$$

↑ Contradiction!

What did we prove,
by the way?

The positive rationals can be placed in a one-to-one correspondence with the positive integers.

The ~~positive~~ rationals can be placed in a one-to-one correspondence with the positive integers.

Exercise

The ~~positive~~ rationals can be placed in a one-to-one correspondence with the integers.

Exercise (easy)

real numbers

The ~~positive rationals~~ can be placed in a one-to-one correspondence with the integers.

True or false?

The last comparison

Set of all numbers
that contain
“5”
in their decimal
expansion

Set of all numbers
that do not contain
“5”
in their decimal
expansion

Set of all numbers
that contain
"5"
in their decimal
expansion

Set of all numbers
that do not contain
"5"
in their decimal
expansion

Which is bigger?

Any guesses?

Well, they are the same.

Well, they are the same.

Exercise

Answers?

Where are all the typical numbers?

- B V Rao

Resonance, Oct 2011 - Volume 16, Number 10
Indian Academy of Sciences


Thank You!